4.2 *One-Sided Limits*

We have made a point of saying that $\lim_{x \to a} f(x) = L$ only if $f(x)$ approaches L as x approaches a from *both* sides of a. For this reason, $\lim_{x \to a} f(x) = L$ is sometimes referred to as the **two-sided limit** of f at a. In some instances it makes sense to talk about limits from one side or the other of a.

Look again at Figure 4.2 which we examined earlier. What can we say about $\lim_{x \to 5} f(x)$. We clearly see that as x approaches 6 from the left (or 'from below'), $f(x)$ approaches 1. And as as x approaches 6 from the right (or 'from above'), $f(x)$ approaches 2.5. So we can say *something* intelligent about f near 6, even though the two-sided limit does not exist. We need some new language to describe this.

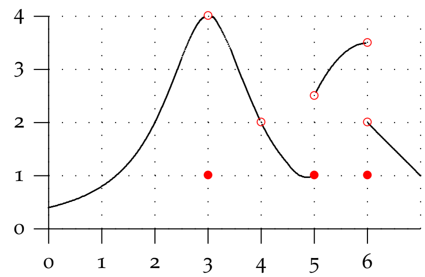

Figure 4.2: What can we say about the limit of f as x approaches 5? 6?

DEFINITION 4.2.1 (One-Sided Limits)**.** Assume that f is defined for all x near a with $x > a$. We write

$$\lim_{x \to a^+} f(x) = L$$

and say that the **limit from the right (above)** of $f(x)$ as x approaches a is L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to but greater than a.

Assume that f is defined for all x near a with $x < a$. We write

$$\lim_{x \to a^-} f(x) = L$$

and say that the **limit from the left (below)** of $f(x)$ as x approaches a is L if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to but smaller than a.

EXAMPLE 4.2.2. Using Figure 4.2 determine the following limits, if they exist.

(*a*) $\lim_{x \to 3^-} f(x)$ (*b*) $\lim_{x \to 3^+} f(x)$ (*c*) $\lim_{x \to 5^-} f(x)$ (*d*) $\lim_{x \to 5^+} f(x)$ (*e*) $\lim_{x \to 6^-} f(x)$ (*f*) $\lim_{x \to 6^+} f(x)$

Solution. Careful: These are all one-sided limits.

(*a*) As x approaches 3 from the left (or below), notice that the values of f approach 4, even though $f(4) = 1$. So $\lim_{x \to 3^-} f(x) = 4$.

(*b*) Similarly, as x approaches 3 from the right (or above), the values of f approach 4, even though $f(4) = 1$. So $\lim_{x \to 3^+} f(x) = 4$. In this case the left- and right-hand limits at 3 were equal.

(*c*) This time, as x approaches 5 from the left (or below), the values of f approach 0.8 so $\lim_{x \to 5^-} f(x) = 1$.

(*d*) However, as x approaches 5 from the right (or above), the values of f approach 2.5 so $\lim_{x \to 5^+} f(x) = 2.5$. The two one-sided limits at $x = 5$ are different. We say that the function $f(x)$ has a **jump** at $x = 5$.

(*e*) Now x approaches 6 from the left (or below), the values of f approach 3.5 so $\lim_{x \to 6^-} f(x) = 3.5$.

(*f*) And as x approaches 6 from the right (or above), the values of f approach 2 so $\lim_{x \to 6^+} f(x) = 2$. The two one-sided limits at $x = 5$ are different. We say that the function $f(x)$ has a **jump** at $x = 5$.

The two points exhibited different behavior. At $x = 3$, both one-sided limits were equal to 4. That's why we were able to say earlier ithat the two-sided limit existed and that $\lim_{x \to 3} f(x) = 4$. On the other hand, because the two one-sided limits are different at 5, this means that f approaches two different numbers as x gets close to 5. Hence the two sided limit there does not exist, that is, $\lim_{x\to 5} f(x)$ DNE. This observation leads to the following theorem.

THEOREM 4.2.3 (One- and Two-Sided Limits). Assume the function f is defined for all x near a, except perhaps at a. Then $\lim_{x\to a} f(x) = L$ if and only if both $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$.

Interpretation. If the two sided limit exists, then so do both one-sided limits and they are both equal to the two-sided limit. If the two one-sided limits are *not equal* or if one or the other does not exist, then the two-sided limit does not exist.

YOU TRY IT 4.5. Using Figure 4.2 determine the following limits, if they exist.

(*a*) $\lim_{x\to 2^-} f(x)$ (*b*) $\lim_{x\to 2^+} f(x)$ (*c*) $\lim_{x\to 2} f(x)$

(*d*) $\lim_{x\to 5} f(x)$ (*e*) $\lim_{x\to 6} f(x)$ (*f*) $\lim_{x\to 4} f(x)$

ANSWER TO YOU TRY IT 4.5. (a-c) 2; (d-e) DNE; (f) 2.

We have stated a number of properties for limits. All of these properties also hold for one-sided limits, as well, with a slight modification for fractional powers.

THEOREM 4.2.4 (One-sided Limit Properties). Limit properties 1 through 9 (the constant multiple, sum, difference, product, quotient, integer power, polynomial, and rational function rules) continue to hold for one-sided limits with the following modification for fractional powers

Assume that m and n are positive integers and that $\frac{n}{m}$ is reduced. Then

(*a*) $\lim_{x\to a^+} [f(x)]^{n/m} = \left[\lim_{x\to a^+} f(x)\right]^{n/m}$ provided that $f(x) \geq 0$ for x near a with $x > a$ if m is even.

(*b*) $\lim_{x\to a^-} [f(x)]^{n/m} = \left[\lim_{x\to a^-} f(x)\right]^{n/m}$ provided that $f(x) \geq 0$ for x near a with $x > a$ if m is even.

EXAMPLE 4.2.5. Here's a simple illustration of this principle. Determine $\lim_{x\to 1^+} \sqrt{2x-2}$. This is the same as $\lim_{x\to 1^+} (2x-2)^{1/2}$. The root is even. Since the limit is from the right, $x > 1$ and so $2x - 2 > 0$. So we may apply the fractional power rule.

$$\lim_{x\to 1^+} (2x-2)^{1/2} \overset{\text{Frac Pwr}}{=} (\lim_{x\to 1^+} 2x-2)^{1/2} \overset{\text{Poly}}{=} 0^{1/2} = 0.$$

On the other hand, if we try to determine $\lim_{x\to 1^-} \sqrt{2x-2}$, since the limit is from the left, $x < 1$ and so $2x - 2 < 0$. So $\sqrt{2x-2}$ is not defined for $x < 1$, and so $\lim_{x\to 1^-} \sqrt{2x-2}$ DNE.

The next few examples illustrate the use of limit properties with piecewise functions.

EXAMPLE 4.2.6. Let $f(x) = \begin{cases} 3x^2+1, & \text{if } x < 2 \\ \sqrt{3x+9} & \text{if } x \geq 2 \end{cases}$. Determine the following limits if they exist.

(*a*) $\lim_{x\to 2^-} f(x)$ (*b*) $\lim_{x\to 2^+} f(x)$ (*c*) $\lim_{x\to 2} f(x)$ (*d*) $\lim_{x\to 0} f(x)$

Solution. We must be careful to use the correct definition of f for each limit.

(*a*) As $x \to 2$ from the left, x is less than 2 so $f(x) = 3x^2 + 1$ there. Thus

$$\lim_{x\to 2^-} f(x) \overset{x\leq 2}{=} \lim_{x\to 2^-} 3x^2+1 \overset{\text{Poly}}{=} 3(-2)^2+1 = 13.$$

(b) As $x \to 2$ from the right, x is greater than 2 so $f(x) = \sqrt{3x+9}$. Thus

$$\lim_{x\to 2^+} f(x) \overset{x \geq 2}{=} \lim_{x\to 2^+} \sqrt{3x+9} \overset{\text{Root}}{=} \sqrt{15}.$$

(c) To determine $\lim_{x\to 2} f(x)$ we compare the one sided limits. Since $\lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$, we conclude that $\lim_{x\to 2} f(x)$ DNE.

(d) To determine $\lim_{x\to 0} f(x)$ we see that the values of x near 0 are less than 2. So $f(x) = 3x^2+1$ there. So

$$\lim_{x\to 0} f(x) \overset{x \leq 2}{=} \lim_{x\to 0} 3x^2+1 \overset{\text{Poly}}{=} 1.$$

We don't need to use the other definition for f since it does not apply to values of x near 0.

EXAMPLE 4.2.7. Let $f(x) = \begin{cases} 3x-1, & \text{if } x \leq 1 \\ x^2+1, & \text{if } 1 < x \leq 5 \\ \frac{x}{x+1} & \text{if } x > 5 \end{cases}$. Determine the following limits if they exist.

(a) $\lim_{x\to 1^-} f(x)$ (b) $\lim_{x\to 1^+} f(x)$ (c) $\lim_{x\to 1} f(x)$

(d) $\lim_{x\to 5^-} f(x)$ (e) $\lim_{x\to 5^+} f(x)$ (f) $\lim_{x\to 5} f(x)$

Solution. We must be careful to use the correct definition of f for each limit. Note how we choose the function!

(a) $\lim_{x\to 1^-} f(x) \overset{x \leq 1}{=} \lim_{x\to 1^-} 3x-1 \overset{\text{Poly}}{=} 2.$

(b) $\lim_{x\to 1^+} f(x) \overset{1 < x \leq 5}{=} \lim_{x\to 1^+} x^2+1 \overset{\text{Poly}}{=} 2.$

(c) Since $\lim_{x\to 1^+} f(x) = 2 = \lim_{x\to 1^-} f(x)$, we conclude that $\lim_{x\to 1} f(x) = 2$.

(d) $\lim_{x\to 5^-} f(x) \overset{1 < x \leq 5}{=} \lim_{x\to 5^-} x^2+1 \overset{\text{Poly}}{=} 26.$

(e) $\lim_{x\to 5^+} f(x) \overset{x \geq 5}{=} \lim_{x\to 5^+} \dfrac{x}{x+1} \overset{\text{Rat'l}}{=} \dfrac{5}{6}.$

(f) Since $\lim_{x\to 5^+} f(x) \neq \lim_{x\to 5^-} f(x)$, we conclude that $\lim_{x\to 5} f(x)$ DNE.

EXAMPLE 4.2.8. Let m be a constant. Let $f(x) = \begin{cases} 3x-m, & \text{if } x \leq 2 \\ mx+9, & \text{if } x > 2 \end{cases}$. For which values of m doest $\lim_{x\to 2} f(x)$ exist?

Solution. Since this is a piecewise function whose definition is split at $x = 2$, we must examine the one-sided limits at 2 and determine whether they are equal. Be careful to use the correct definition of f for each limit. Note how we choose the function! From the left,

$$\lim_{x\to 2^-} f(x) \overset{x \leq 2}{=} \lim_{x\to 2^-} 3x-m \overset{\text{Poly}}{=} 6-m.$$

From the right,

$$\lim_{x\to 2^+} f(x) \overset{x \geq 2}{=} \lim_{x\to 2^+} mx+9 \overset{\text{Poly}}{=} 2m+9.$$

We need $\lim_{x\to 2^+} f(x) = 2 = \lim_{x\to 2^-} f(x)$ for $\lim_{x\to 2} f(x)$ to exist. So we need

$$6-m = 2m+9, \text{ or, } -3 = 3m, \text{ so } m = -1.$$

You should now check that both limits agree if $m = -1$.

4.3 *Most Limits Are Not Simple*

Let's return to the original motivation for calculating limits. We were interested in finding the 'slope' of a curve and this led to looking at limits that have the form

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}.$$

Assuming that f is continuous, this limit cannot be evaluated by any of the basic limit properties since the denominator is approaching 0. More specifically, as $x\to a$, this difference quotient has the **indeterminate form** $\frac{0}{0}$. To evaluate this limit we must do more work. Let's look at an

EXAMPLE 4.3.1. Let $f(x)=x^2-3x+1$. Determine the slope of this curve right at $x=4$.

Solution. To find the slope of a curve we must evaluate the difference quotient

$$\lim_{x\to 4}\frac{f(x)-f(4)}{x-4}=\lim_{x\to 4}\frac{(x^2-3x+1)-5}{x-4}.$$

Though this is a rational function, the limit properties do not apply since the denominator is 0 at 4, and so is the numerator (check it!). Instead, we must 'do more work.'

$$\lim_{x\to 4}\frac{f(x)-f(4)}{x-4}=\lim_{x\to 4}\frac{x^2-3x-4}{x-4}=\lim_{x\to 4}\frac{(x-4)(x+1)}{x-4}=\lim_{x\to 4}x+1\overset{\text{Poly}}{=}5.$$

Only at the very last step were we able to use a limit property.

The Indeterminate Form $\frac{0}{0}$

Many of the most important limits we will see in the course have the indeterminate form $\frac{0}{0}$ as in the previous example. To evaluate such limits, if they exist, requires 'more work' typically of the following type.

- factoring
- using conjugates[1]
- simplifying
- making use of known limits

[1] Recall that if $a>0$, then $\sqrt{a}+b$ and $\sqrt{a}-b$ are called **conjugates**. Notice that

$$(\sqrt{a}+b)(\sqrt{a}-b)=a-b^2.$$

There is no middle term.

Let's look at some examples of each.

EXAMPLE 4.3.2 (Factoring). Factoring is one of the most critical tools in evaluating the sorts of limits that arise in elementary calculus. Evaluate $\lim_{x\to 2}\frac{2x^2-6x+4}{x^2+2x-8}$.

Solution. Notice that this limit has the indeterminate form $\frac{0}{0}$. Factoring is the key.

$$\lim_{x\to 2}\frac{2x^2-6x+4^{\nearrow 0}}{x^2+2x-8_{\searrow 0}}=\lim_{x\to 2}\frac{2(x-1)(x-2)}{(x+4)(x-2)}=\lim_{x\to 2}\frac{2(x-1)}{x+4}\overset{\text{Rational}}{=}\frac{2}{6}=\frac{1}{3}.$$

Only at the very last step were we able to use a limit property.

EXAMPLE 4.3.3 (Factoring). Evaluate $\lim_{x\to -2}\frac{x^2+8x+12}{x^3+2x^2}$.

Solution. This limit has the indeterminate form $\frac{0}{0}$. Factoring is the key.

$$\lim_{x\to-2} \frac{x^2+8x+12^{\nearrow 0}}{x^3+2x^2_{\searrow 0}} = \lim_{x\to-2} \frac{(x+6)(x+2)}{x^2(x+2)} = \lim_{x\to-2} \frac{(x+6)}{x^2} \overset{\text{Rat'l}}{=} \frac{4}{4} = 1.$$

Only at the very last step were we able to use a limit property.

EXAMPLE 4.3.4. (Conjugates) Evaluate $\lim\limits_{x\to 4} \dfrac{\sqrt{x}-2}{2x-8}$.

Solution. Notice that this limit has the indeterminate form $\frac{0}{0}$. Let's see how conjugates help.

$$\begin{aligned}
\lim_{x\to 4} \frac{\sqrt{x}-2^{\nearrow 0}}{2x-8_{\searrow 0}} = \lim_{x\to 4} \frac{\sqrt{x}-2}{2x-8} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} &= \lim_{x\to 4} \frac{x-4}{(2x-8)(\sqrt{x}+2)} \\
&= \lim_{x\to 4} \frac{x-4}{2(x-4)(\sqrt{x}+2)} \\
&= \lim_{x\to 4} \frac{1}{2(\sqrt{x}+2)} \overset{\text{Root}}{=} \frac{1}{2\sqrt{4}+2} = \frac{1}{8}.
\end{aligned}$$

EXAMPLE 4.3.5. (Conjugates) Here's another: Evaluate $\lim\limits_{x\to 2} \dfrac{\sqrt{x+4}-\sqrt{6}}{x-2}$.

Solution. Notice that this limit has the indeterminate form $\frac{0}{0}$. Use conjugates again.

$$\begin{aligned}
\lim_{x\to 2} \frac{\sqrt{x+4}-\sqrt{6}^{\nearrow 0}}{x-2_{\searrow 0}} &= \lim_{x\to 2} \frac{\sqrt{x+4}-\sqrt{6}}{x-2} \cdot \frac{\sqrt{x+4}+\sqrt{6}}{\sqrt{x+4}+\sqrt{6}} \\
&= \lim_{x\to 2} \frac{(x+4)-6}{(x-2)(\sqrt{x+4}+\sqrt{6})} \\
&= \lim_{x\to 2} \frac{x-2}{(x-2)(\sqrt{x+4}+\sqrt{6})} \\
&= \lim_{x\to 2} \frac{1}{\sqrt{x+4}+\sqrt{6}} \\
&\overset{\text{Root}}{=} \frac{1}{2\sqrt{6}}.
\end{aligned}$$

EXAMPLE 4.3.6. (Conjugates) Evaluate $\lim\limits_{x\to 1} \dfrac{x^2-1}{\sqrt{x+3}-2}$.

Solution. This limit has the indeterminate form $\frac{0}{0}$.

$$\begin{aligned}
\lim_{x\to 1} \frac{x^2-1^{\nearrow 0}}{\sqrt{x+3}-2_{\searrow 0}} &= \lim_{x\to 1} \frac{x^2-1}{\sqrt{x+3}-2} \cdot \frac{\sqrt{x+3}+2}{\sqrt{x+3}+2} \\
&= \lim_{x\to 1} \frac{(x^2-1)(\sqrt{x+3}+2)}{(x+3)-4} \\
&= \lim_{x\to 1} \frac{(x-1)(x+1)(\sqrt{x+3}+2)}{x-1} \\
&= \lim_{x\to 1} (x+1)(\sqrt{x+3}+2) \\
&\overset{\text{Prod, Root}}{=} 2(2+2) = 8.
\end{aligned}$$

EXAMPLE 4.3.7 (Simplification). Sometimes limits, like this next one, involve compound fractions. One method of attack is to carefully simplify them. Evaluate $\lim_{x\to 2} \frac{\frac{2}{x+1} - \frac{2}{x^2-1}}{x-2}$.

Solution. Notice that this limit has the indeterminate form $\frac{0}{0}$. Use common denominators to simplify.

$$\begin{aligned}
\lim_{x\to 2} \frac{\frac{2}{x+1} - \frac{2}{x^2-1}\nearrow^{0}}{x-2_{\searrow 0}} &= \lim_{x\to 2} \frac{\frac{2(x-1)-2}{(x+1)(x-1)}}{x-2} \\
&= \lim_{x\to 2} \frac{2x-4\nearrow^{0}}{(x+1)(x-1)(x-2)_{\searrow 0}} \\
&= \lim_{x\to 2} \frac{2}{(x+1)(x-1)} \overset{\text{Rational}}{=} \frac{2}{3}.
\end{aligned}$$

EXAMPLE 4.3.8 (Simplification). Evaluate $\lim_{x\to 1} \frac{\frac{1}{x+1} - \frac{1}{2}}{x-1}$.

Solution. Notice that this limit has the indeterminate form $\frac{0}{0}$. Use common denominators to simplify.

$$\lim_{x\to 1} \frac{\frac{1}{x+1} - \frac{1}{2}\nearrow^{0}}{x-1_{\searrow 0}} = \lim_{x\to 1} \frac{\frac{2-(x+1)}{2(x+1)}}{x-1} = \lim_{x\to 2} \frac{1-x\nearrow^{0}}{2(x+1)(x-1)_{\searrow 0}} = \lim_{x\to 2} \frac{-1}{2(x+1)} \overset{\text{Rat'l}}{=} \frac{1}{2}.$$

EXAMPLE 4.3.9 (Simplification). Evaluate $\lim_{h\to 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h}$.

Solution. Notice that this limit has the indeterminate form $\frac{0}{0}$.

$$\lim_{h\to 0} \frac{\frac{2}{x+h} - \frac{2}{x}\nearrow^{0}}{h_{\searrow 0}} = \lim_{h\to 0} \frac{\frac{2x-2(x+h)}{(x+h)(x)}}{h} = \lim_{h\to 0} \frac{-2h}{(x+h)(x)(h)} = \lim_{h\to 0} \frac{-2}{(x+h)(x)} \overset{\text{Rat'l}}{=} -\frac{2}{x^2}.$$

EXAMPLE 4.3.10 (Simplification). Evaluate $\lim_{x\to 1} \frac{\frac{4}{x^2+7} - \frac{1}{2}}{x-1}$.

Solution. Notice that this limit has the indeterminate form $\frac{0}{0}$. Use common denominators to simplify.

$$\begin{aligned}
\lim_{x\to 1} \frac{\frac{4}{x^2+7} - \frac{1}{2}}{x-1} = \lim_{x\to 1} \frac{\frac{8-(x^2+7)}{2(x^2+7)}}{x-1} = \lim_{x\to 1} \frac{\frac{1-x^2}{2(x^2+7)}}{x-1} &= \lim_{x\to 1} \frac{(1-x)(1+x)}{2(x^2+7)(x-1)} \\
&= \lim_{x\to 1} \frac{-(1+x)}{2(x^2+7)} \\
&\overset{\text{Rat'l}}{=} \frac{-2}{16} = -\frac{1}{8}.
\end{aligned}$$

4.4 *Practice Problems*

EXAMPLE 4.4.1 (Simplification). Evaluate $\lim_{x\to 4} \frac{x^2-6x+8}{x-4}$.

Solution. Notice that this limit has the indeterminate form $\frac{0}{0}$. Use factoring to simplify this rational function.

$$\lim_{x\to 4}\frac{x^2-6x+8}{x-4}=\lim_{x\to 4}\frac{(x-4)(x-2)}{x-4}=\lim_{x\to 4}x-2\overset{\text{Linear}}{=}2.$$

EXAMPLE 4.4.2 (Simplification). Evaluate $\lim_{x\to -2}\frac{x+2}{4-x^2}$.

Solution. Notice that this limit has the indeterminate form $\frac{0}{0}$. Use factoring to simplify this rational function.

$$\lim_{x\to -2}\frac{x+2}{4-x^2}=\lim_{x\to -2}\frac{x+2}{(2-x)(2+x)}=\lim_{x\to -2}\frac{1}{2-x}\overset{\text{Rat'l}}{=}\frac{1}{4}.$$

EXAMPLE 4.4.3 (Simplification). Evaluate $\lim_{x\to 5}\frac{x^2-3x-10}{x^2-25}$.

Solution. Notice that this limit has the indeterminate form $\frac{0}{0}$. Use factoring to simplify this rational function.

$$\lim_{x\to 5}\frac{x^2-3x-10}{x^2-25}=\lim_{x\to 5}\frac{(x+2)(x-5)}{(x-5)(x+5)}=\lim_{x\to 5}\frac{x+2}{x+5}\overset{\text{Rat'l}}{=}\frac{7}{10}.$$

EXAMPLE 4.4.4 (Simplification). Evaluate $\lim_{x\to -1}\frac{x^3-x}{x^2-5x-6}$.

Solution. Notice that this limit has the indeterminate form $\frac{0}{0}$. Use factoring to simplify this rational function.

$$\lim_{x\to -1}\frac{x^3-x}{x^2-5x-6}=\lim_{x\to -1}\frac{x(x^2-1)}{(x+1)(x-6)}=\lim_{x\to -1}\frac{x(x-1)(x+1)}{(x+1)(x-6)}=\lim_{x\to -1}\frac{x(x-1)}{x-6}$$
$$\overset{\text{Rat'l}}{=}-\frac{2}{7}.$$

EXAMPLE 4.4.5 (Simplification). Evaluate $\lim_{x\to 0}\frac{\frac{3}{2x+1}-3}{x}$.

Solution. Notice that this limit has the indeterminate form $\frac{0}{0}$. Use common denominators to simplify.

$$\lim_{x\to 0}\frac{\frac{3}{2x+1}-3}{x}=\lim_{x\to 0}\frac{\frac{3-3(2x+1)}{2x+1}}{x}=\lim_{x\to 0}\frac{\frac{3-6x+3}{2x+1}}{x}=\lim_{x\to 0}\frac{-6x}{x(2x+1)}=\lim_{x\to 2}\frac{-6}{2x+1}\overset{\text{Rat'l}}{=}-6.$$

EXAMPLE 4.4.6 (Simplification). Evaluate $\lim_{x\to 2}\frac{\frac{1}{x^2}-\frac{1}{4}}{2-x}$.

Solution. Notice that this limit has the indeterminate form $\frac{0}{0}$. Use common denominators to simplify.

$$\lim_{x\to 2}\frac{\frac{1}{x^2}-\frac{1}{4}}{2-x}=\lim_{x\to 2}\frac{\frac{4-x^2}{4x^2}}{2-x}=\lim_{x\to 2}\frac{4-x^2}{(4x^2)(2-x)}=\lim_{x\to 2}\frac{(2-x)(2+x)}{(4x^2)(2-x)}=\lim_{x\to 2}\frac{2+x}{4x^2}$$
$$\overset{\text{Rat'l}}{=}\frac{4}{16}=\frac{1}{4}.$$

EXAMPLE 4.4.7 (Simplification). Evaluate $\lim_{x\to 1} \frac{\frac{1}{x^2+1}-\frac{1}{2}}{x-1}$.

Solution. Notice that this limit has the indeterminate form $\frac{0}{0}$. Use common denominators to simplify.

$$\begin{aligned}\lim_{x\to 1} \frac{\frac{1}{x^2+1}-\frac{1}{2}}{x-1} = \lim_{x\to 1} \frac{\frac{2-x^2-1}{2(x^2+1)}}{x-1} = \lim_{x\to 1} \frac{\frac{1-x^2}{2(x^2+1)}}{x-1} &= \lim_{x\to 1} \frac{1-x^2}{2(x^2+1)(x-1)} \\ &= \lim_{x\to 1} \frac{(1-x)(1+x)}{2(x^2+1)(x-1)} \\ &= \lim_{x\to 1} \frac{-(1+x)}{2(x^2+1)} \\ &\overset{\text{Rat'l}}{=} \frac{-2}{4} = -\frac{1}{2}.\end{aligned}$$

EXAMPLE 4.4.8 (Simplification). Evaluate $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$.

Solution. Notice that this limit has the indeterminate form $\frac{0}{0}$. Use conjugates to simplify.

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1} = \lim_{x\to 1} \frac{x-1}{(x-1)(\sqrt{x}+1)} = \lim_{x\to 1} \frac{1}{\sqrt{x}+1} \overset{\text{Root}}{=} \frac{1}{2}.$$

EXAMPLE 4.4.9 (Simplification). Evaluate $\lim_{x\to 3} \frac{x-3}{\sqrt{x+1}-2}$.

Solution. Notice that this limit has the indeterminate form $\frac{0}{0}$. Use conjugates to simplify.

$$\begin{aligned}\lim_{x\to 3} \frac{x-3}{\sqrt{x+1}-2} = \lim_{x\to 3} \frac{x-3}{\sqrt{x+1}-2} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} &= \lim_{x\to 3} \frac{(x-3)(\sqrt{x+1}+2)}{x+1-4} \\ &= \lim_{x\to 3} \frac{(x-3)(\sqrt{x+1}+2)}{x-3} \\ &= \lim_{x\to 3} \sqrt{x+1}+2 \overset{\text{Root}}{=} 4.\end{aligned}$$

EXAMPLE 4.4.10 (Simplification). Evaluate $\lim_{x\to 0} \frac{\sqrt{4-x}-2}{x^2-x}$.

Solution. Notice that this limit has the indeterminate form $\frac{0}{0}$. Use conjugates to simplify.

$$\begin{aligned}\lim_{x\to 0} \frac{\sqrt{4-x}-2}{x^2-x} = \lim_{x\to 0} \frac{\sqrt{4-x}-2}{x^2-x} \cdot \frac{\sqrt{4-x}+2}{\sqrt{4-x}+2} &= \lim_{x\to 0} \frac{(4-x)-4}{(x^2-x)(\sqrt{4-x}+2)} \\ &= \lim_{x\to 0} \frac{-x}{(x^2-x)(\sqrt{4-x}+2)} \\ &= \lim_{x\to 0} \frac{-x}{x(x-1)(\sqrt{4-x}+2)} \\ &= \lim_{x\to 0} \frac{-1}{(x-1)(\sqrt{4-x}+2)} \\ &\overset{\text{Root, Prod}}{=} \frac{-1}{-4} = \frac{1}{4}.\end{aligned}$$