Continuity in More Detail

5-Minute Review: Continuity

We have worked off and on with continuous functions. Recall

DEFINITION 8.1.1 (Continuity at a Point). A function $f(x)$ is **continuous** at a if $\lim_{x \to a} f(x) = f(a)$. If f is not continuous at a, then a is a point of **discontinuity**.

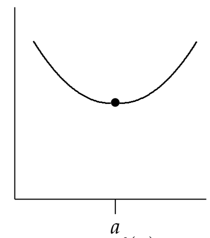


Figure 8.1: $f(x)$ is continuous at $x = a$; it is "unbroken" there.

This definition can be broken down to three facts that must be verified for f to be continuous at a. It is usually easiest to use the list below in problems.

Continuity Checklist. f is continuous at a if the following three conditions hold:

1. $f(a)$ is defined (i.e., a is in the domain of f).
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$.

For any polynomial $p(x)$, we know that $\lim_{x \to a} p(x) = p(a)$ so polynomials are continuous at all points. However, for a general function f it need not be the case that $\lim_{x \to a} f(x) = f(a)$. This can fail to happen in several ways.

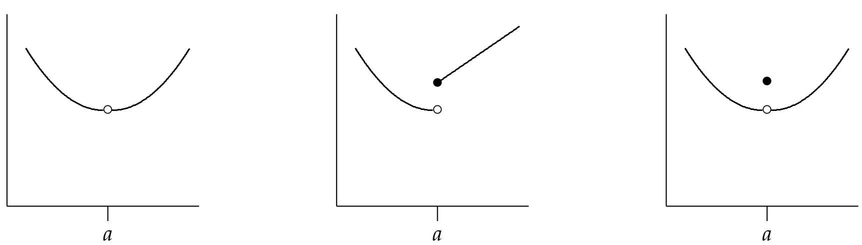


Figure 8.2: Three of the ways in which a function f may fail to be continuous because $\lim_{x \to a} f(x) \neq f(a)$.

1. $f(a)$ may not be defined;
2. $\lim_{x \to a} f(x)$ may not exist;
3. $\lim_{x \to a} f(x)$ and $f(a)$ may both exist but not be equal.

Intuition and Extensions. Because we have left- and right-hand limits we can define left and right continuous functions.

DEFINITION 8.1.2. We say that

- $f(x)$ is **continuous from the right at** $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$.
- $f(x)$ is **continuous from the left at** $x = a$ if $\lim_{x \to a^-} f(x) = f(a)$.
- So, if $f(x)$ is both left and right continuous at $x = a$, then it is continuous.

YOU TRY IT 8.1. Which graph, if any, in Figure 8.2 is continuous from the left at a? Right?

ANSWER TO YOU TRY IT 8.1. The middle graph is continuous from the right at a because $\lim_{x \to a^+} f(x) = f(a)$.

Let's compare the current definitions with one we saw earlier.

DEFINITION 8.1.3 (Removable Discontinuity). A function f has a **removable discontinuity** (RD) at a if the following hold:

(1) $\lim_{x\to a} f(x)$ exists (and is finite).

(2) $\lim_{x\to a} f(x) \neq f(a)$. Note: $f(a)$ may not even exist.

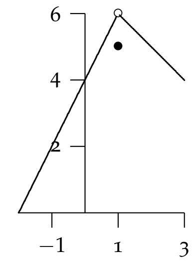


Figure 8.3: f is *not* continuous at $x = 1$. It has a *removable discontinuity* there. See Definition 8.1.3.

Remember that f is continuous at a if $\lim_{x\to a} f(x) = f(a)$. So condition 2 in the definition ensures that f is NOT continuous at a. On the other hand, the function is well-behaved near a, since $\lim_{x\to a} f(x)$ exists. In fact, if we defined (or redefined) $f(a)$ to be $\lim_{x\to a} f(x)$, then f would be continuous. That is, we could *remove the discontinuity* by redefining f and filling in the hole in the graph (see Figure 8.3).

YOU TRY IT 8.2. Which graph, if any, in Figure 8.2 on page 1 has a removable discontinuity

ANSWER TO YOU TRY IT 8.2. The righthand graph is has a removable discontinuity at a because $\lim_{x\to a^+} f(x)$ exists but does not equal $f(a)$.

YOU TRY IT 8.3. Using the information given, fill in the remaining table entries. For a few of the numerical answers, there may be more than one correct answer. We will do this in class.

a	$\lim_{x\to a^-} f(x)$	$\lim_{x\to a^+} f(x)$	$\lim_{x\to a} f(x)$	$f(a)$	Left Cont	Right Cont	Cont	Rem Disc
-5	4	2		2				
-4			4	4				
-3	1	2		3				
-2			-3				Cont	
-1			DNE	3		Yes		
0		6			Yes	Yes		
1		3		4	Yes			
2	2	2			No			
3		-1				Yes	No	
4	5	-5			Yes			
5			4				Yes	
6				7			Yes	
7	3						No	Yes
8				1			No	Yes

Functions We Know Are Continuous

In our study of limits we remarked that polynomials are continuous everywhere since $\lim_{x\to a} p(x) = p(a)$ for any polynomial p and any number a. Using the quotient rule for limits, we then saw that a rational function r is continuous at each point in its domain, since $\lim_{x\to a} r(x) = r(a)$ for any a at which r is defined.

THEOREM 8.1.4. (Polynomials and Rational Functions Are Continuous)

1. A polynomial is continuous everywhere, i.e. for all x.
2. A rational functions $r(x) = \frac{p(x)}{q(x)}$ where p and q are polynomials is continuous at all points in its domain, i.e., where $q(x) \neq 0$. (Note: Where a rational function is not continuous, it may have either a removable discontinuity or a vertical asymptote.)

EXAMPLE 8.1.5. Let $p(x) = 2x^2 + 5x - 11$. Since $p(x)$ is a polynomial, we know $\lim_{x\to a} p(x) = p(a)$ for all real numbers a.

Let $r(x) = \frac{2x^2+5x-11}{x^2-2x}$. Since $r(x)$ is rational, we know $\lim_{x\to a} r(x) = r(a)$ for all points in its domain. So it is continuous for all $x \neq 0, 2$.

Take a moment to write down other types of functions that you know are continuous on their domains (e.g., root functions). We will see several more 'types' of functions that are continuous in the coming days. Knowing that a function is continuous makes limit calculations trivial: If $f(x)$ is continuous at a, to evaluate $\lim_{x\to a} f(x)$ all we need to do is evaluate f at a, that is, $\lim_{x\to a} f(x) = f(a)$. No other work is required. This is what makes them important. There are no surprises!

Determining Where Functions Are Continuous Algebraically

EXAMPLE 8.1.6. Determine whether the following functions are continuous at the given points.

(*a*) $r(x) = \dfrac{x^2-1}{x^2-4x+3}$ at $a = 1$, 2, and 3.

(*b*) $g(x) = \begin{cases} x+2, \text{if } x \geq 3 \\ x^2+1, \text{if } x < 3 \end{cases}$ at $a = 0$ and 3.

Solution.

(*a*) $r(x)$ is a rational function. A rational function is continuous at every point in its domain. Notice that

$$r(x) = \frac{x^2-1}{x^2-4x+3} = \frac{(x-1)(x+1)}{(x-1)(x-3)}$$

is not defined at $x = 1$ and $x = 3$, so r is not continuous at either of these points. However, since $x = 2$ is in the domain of r, then r is continuous at $x = 2$ and, in fact, at every real number not equal to 1 or 3. In fact it is easy to calculate

$$\lim_{x\to 2} \frac{x^2-1}{x^2-4x+3} = \frac{4-1}{4-8+3} = -3 = f(2).$$

(*b*) $g(x)$ is a piecewise function and the definition of g changes at $x = 3$. Use the checklist.

(*1*) $g(3) = 3 + 2 = 5$.

(*2*) To determine $\lim_{x\to 3} g(x)$ use one-sided limits since the definition of g is different on either side of 3.

$$\lim_{x\to 3^+} g(x) = \lim_{x\to 3^+} x + 2 = 5$$

while

$$\lim_{x\to 3^-} g(x) = \lim_{x\to 3^-} 3^2 + 1 = 10.$$

Since the two one-sided limits differ, $\lim_{x\to 3} g(x)$ DNE. So g is not continuous at $x = 3$.

(*c*) What about g at $x = 0$? Since 0 is less than 3, $g(0) = 0^2 + 1 = 1$. Further, near 0 on either side, $g(x) = x^2 + 1$ so $\lim_{x\to 0} g(x) = \lim_{x\to 0} x^2 + 1 = 1 = g(0)$. So g is continuous at $x = 0$.

EXAMPLE 8.1.7. Let $g(x) = \begin{cases} x^2+m, \text{if } x \leq 2 \\ mx+7, \text{if } x > 2 \end{cases}$, where m is a constant. Is there any value of m for which g would be continuous at $x = 2$?

Solution. $g(x)$ is a piecewise function and the definition of g changes at $x = 2$.

(*1*) $g(2) = 2^2 + m$.

(2) To find $\lim_{x\to 2} g(x)$ we must use one-sided limits since the definition of g is different on either side of 2.

$$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} mx + 7 = 2m + 7$$

while

$$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} x^2 + m = 4 + m.$$

We need the two one-sided limits to be equal: $2m + 7 = 4 + m$ so $m = -3$. If $m = -3$, then

$$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} -3x + 7 = 1$$

and

$$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} x^2 - 3 = 1.$$

So $\lim_{x\to 2} g(x) = 1$.

(3) Now with $m = -3$, $g(2) = 2^2 - 3 = 1$ also. So g is continuous at 2 if $m = -3$.

EXAMPLE 8.1.8. Let $f(x) = \begin{cases} 2x+4, \text{if } x < 1 \\ 5, \text{if } x = 1 \\ -x+7, \text{if } x > 1 \end{cases}$. Is f continuous at $x = 0$? At $x = 1$?

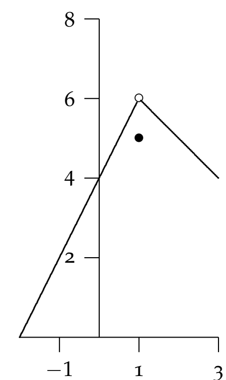


Figure 8.4: f is *not* continuous at $x = 1$. It has a *removable discontinuity* there. See Definition 8.1.3.

Solution. At $x = 0$: Use the continuity checklist.

(1) Since $0 < 1$, we have $f(0) = 2(0) + 4 = 4$.

(2) $\lim_{x\to 0} f(x) \overset{x \leq 1}{=} \lim_{x\to 0} 2x + 4 = 4$,

(3) So $f(0) = \lim_{x\to 0} f(x)$ and f is continuous at 0

At $x = 1$ the definition of f changes.

(1) $f(1) = 5$.

(2) To find $\lim_{x\to 1} f(x)$ we must use one-sided limits since the definition of f is different on either side of 1.

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} = -x + 7 = 6$$

while

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2x + 4 = 6.$$

So $\lim_{x\to 1} f(x) = 6$.

(3) Finally, $f(1) \neq \lim_{x\to 1} f(x)$ also. So f is not continuous at 1.

Removable Discontinuities, Algebraically

In Example 8.1.8 just above, even though f was not continuous at $x = 1$, the behavior of f *near* 1 was reasonable. The limit existed (and was finite). The problem was that the limit value and the function value were different. Recall

DEFINITION 8.1.9 (Removable Discontinuity)**.** A function f has a **removable discontinuity** (RD) at a if the following hold:

(1) $\lim_{x\to a} f(x)$ exists (and is finite).

(2) $\lim_{x\to a} f(x) \neq f(a)$. Note: $f(a)$ may not even exist.

As we said earlier, f is continuous at a if $\lim_{x\to a} f(x) = f(a)$. So condition 2 in the definition ensures that f is NOT continuous at a. On the other hand, the function is well-behaved near a, since $\lim_{x\to a} f(x)$ exists. In fact, if we redefined $f(a)$ to be $\lim_{x\to a} f(x)$, then f would be continuous. That is, we could *remove the discontinuity* by redefining f and filling in the hole in the graph (see Figure 8.4).

The next example is more typical of where we see removable discontinuities.

EXAMPLE 8.1.10. Determine the points at which $f(x) = \dfrac{x^2-5x+6}{x^3+x^2-12x}$ is discontinuous. At which points does f have VA's? Removable discontinuities?

Solution. Since $f(x)$ is rational it is continuous at all points in its domain. So it will fail to be continuous where the denominator is equal to 0. So let's factor f:

$$f(x) = \frac{x^2-5x+6}{x^3+x^2-12x} = \frac{(x-2)(x-3)}{x(x+4)(x-3)}, \qquad x \neq -4, 0, 3.$$

f is discontinuous at -4, 0, and 3. Now examine appropriate limits to check for VA's and removable discontinuities. (Can you predict which are which?)

At $x = -4$:

$$\lim_{x\to -4^-} f(x) = \lim_{x\to -4^-} \frac{(x-2)(x-3)}{x(x+4)(x-3)} = \lim_{x\to -4^-} \frac{\overbrace{x-2}^{\to -6}}{\underbrace{x(x+4)}_{\to -4\cdot 0^- = 0^+}} = -\infty.$$

This is enough to conclude that f has a VA at -4. Caution: Take care with the calculation of the sign in the denominator.

At $x = 0$:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{(x-2)(x-3)}{x(x+4)(x-3)} = \lim_{x\to 0^-} \frac{\overbrace{x-2}^{\to -2}}{\underbrace{x(x+4)}_{\to 0^-\cdot 4 = 0^-}} = -\infty.$$

This is enough to conclude that f has a VA at 0. Again: Take care with the calculation of the sign in the denominator.

At $x = 3$: Having seen the factorization of f, we know that we can calculate a two-sided limit at 3.

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{(x-2)(x-3)}{x(x+4)(x-3)} = \lim_{x\to 3} \frac{x-2}{x(x+4)} = \frac{1}{21}.$$

Since $\lim_{x\to 3} f(x)$ exists but $f(3)$ is not defined, then f has a removable discontinuity at $x = 3$.

YOU TRY IT 8.4. Determine $\lim_{x\to -4^+} f(x)$ and $\lim_{x\to 0^+} f(x)$ for the function in Example 8.1.10.

Answer to YOU TRY IT 8.4 : Both are ∞.

EXAMPLE 8.1.11. Determine the points at which $f(x) = \dfrac{\frac{1}{x-2} - \frac{1}{2}}{x-4}$ is discontinuous. At which points does f have VA's? Removable discontinuities?

Solution. Since $f(x)$ is rational it is continuous at all points in its domain. We can see immediately that f is not defined at $x = 4$ and $x = 2$, where there would be division by 0, so f is not continuous at these two points. Let's simplify the expression for f before taking the appropriate limits.

$$f(x) = \frac{\frac{1}{x-2} - \frac{1}{2}}{x-4} = \frac{\frac{2-(x-2)}{2(x-2)}}{x-4} = \frac{4-x}{2(x-2)(x-4)}.$$

At $x = 2$:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{4-x}{2(x-2)(x-4)} = \lim_{x\to 2^-} \frac{-1}{2(x-2)} = \lim_{x\to 2^-} \frac{-1}{\underbrace{2(x-2)}_{\to 2\cdot 0^- = 0^-}} = \infty.$$

This is enough to conclude that f has a VA at 2.

At $x = 4$: Having seen the factorization of f, we know that we can calculate a two-sided limit at 4.

$$\lim_{x\to 4} f(x) = \lim_{x\to 4} \frac{4-x}{2(x-2)(x-4)} = \lim_{x\to 4} \frac{-1}{2(x-2)} = -\frac{1}{8}.$$

Since $\lim_{x\to 4} f(x)$ exists but $f(4)$ is not defined, then f has a removable discontinuity at $x = 4$.

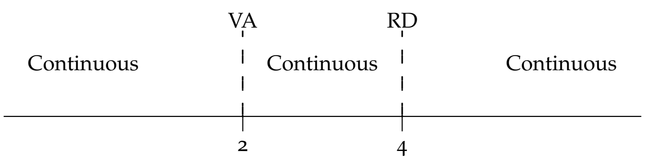


Figure 8.5: A nice way to summarize all the information about f that we have found is with a number line. In this case, $f(x)$ is continuous on $(-\infty, 2)$ and $(2, 4)$, and $(4, \infty)$.

YOU TRY IT 8.5. Determine $\lim_{x\to 2^+} f(x)$ for the function in Example 8.1.11.

Answer to YOU TRY IT 8.5 : $-\infty$.

YOU TRY IT 8.6. Determine where the function $f(x) = \dfrac{x^2-1}{x^2-3x+2}$ has vertical asymptotes and where it has removable discontinuities.

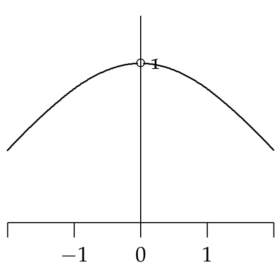
Answer to YOU TRY IT 8.6 : VA at $x = 2$. Removable discontinuity at $x = 1$ since $\lim_{x\to 1} f(x) = -2$ and $f(a)$ UND.

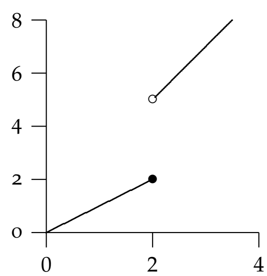
YOU TRY IT 8.7. Consider the two graphs below that we saw earlier this term when first considering limits. Discuss the type of discontinuity in each.

Answer to YOU TRY IT 8.7 : Since $\lim_{x\to 0} \frac{\sin x}{x} = 1$ but $\frac{\sin x}{x}$ is not defined at 0, then by Definition **??** there is a removable discontinuity at 0.

On the right $\lim_{x\to 2} f(x)$ DNE because the two one-sided derivatives are not equal. So the discontinuity at 2 is not removable.

$\dfrac{\sin x}{x}$

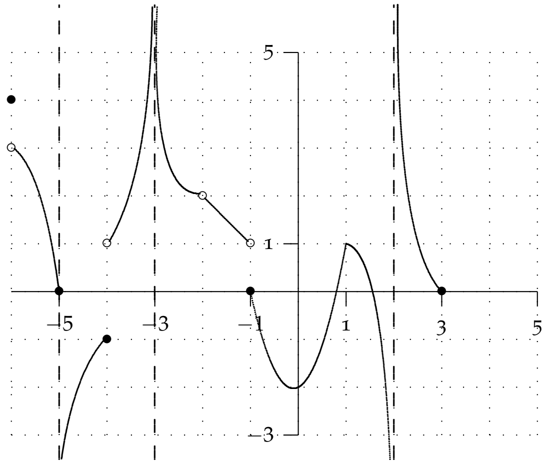




$$f(x) = \begin{cases} x, & \text{if } x \le 2 \\ 2x+1, & \text{if } x > 2 \end{cases}$$

Problems

1. Use the graph of f to evaluate each of the expressions in the table or explain why the value does not exist. For the "Cont(inuity)" column, use "Left", "Right", or "Cont" if it is continuous, and "RD" if it has a removable discontinuity, and "No" if none of these hold. For the last few points, complete the graph so that the given information is true. (When the graph goes off the grid, the function is becoming unbounded.)



a	$\lim_{x\to a^-} f(x)$	$\lim_{x\to a^+} f(x)$	$\lim_{x\to a} f(x)$	$f(a)$	Cont	VA
-5						
-4						
-3						
-2						
-1						
0						
1						
2						
3					C	
4				5		Yes

2. (*a*) True or false: Even if $f(x)$ has a removable discontinuity at $x = a$ it still might have a VA at $x = a$. Explain.

(*b*) Give the equation of a rational function with a VA at $x = -2$ and a removable discontinuity at $x = 6$.

3. Here are five straightforward one-sided limits problems. Use $+\infty$ or $-\infty$ if appropriate. When the denominator goes to 0 but the numerator does not, determine the signs of each and then determine the limit. For indeterminate $\frac{0}{0}$ limits do more work.

(*a*) $\displaystyle\lim_{x\to 3^+} \frac{-2x+1}{x-3}$ (*b*) $\displaystyle\lim_{x\to -4^-} \frac{x^2+3x-4}{x+4}$

(*c*) $\displaystyle\lim_{x\to -2^+} \frac{x^2-4}{x+4}$ (*d*) $\displaystyle\lim_{x\to 1^+} \frac{x^2}{x^2-1}$

(*e*) $\displaystyle\lim_{x\to 2^-} \frac{x^2}{x(x-2)}$

4. Which of the functions above have a VA at the point in the limit? How can you tell?

5. This last problem is more like a test question.

(*a*) $f(x) = \dfrac{2x-8}{x^2-4x}$ is continuous except at two points: $x =$ ______ , because f is ________________ .

(*b*) At each point from part (a), determine the limit. If infinite limits are required check both $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$.

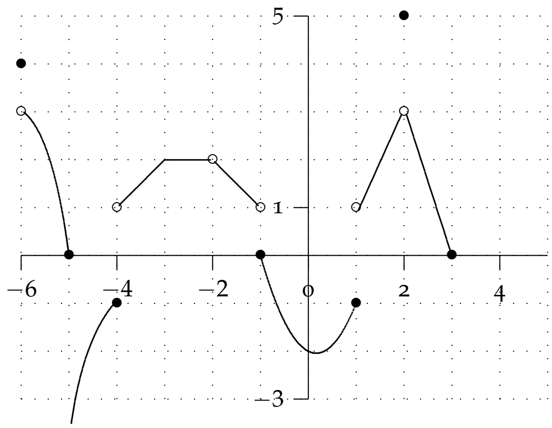
(*c*) Does f have a VA at either point? Explain.

(*d*) Does f have a removable discontinuity (RD) at either point? Explain.

6. Determine where each of these functions is continuous. Does either have any VA's? Removable discontinuities. Show your work.

(*a*) $g(x) = \dfrac{x^2-2x-8}{x^2-16}$ (*b*) $f(x) = \dfrac{\frac{1}{x-4}+\frac{1}{2}}{x-2}$

7. Use the graph of f to evaluate each of the expressions in the table or explain why the value does not exist. For the "Cont(inuity)" column, your answer should be: "Left", "Right", or "Cont" if it is continuous, and "RD" if it has a removable discontinuity, and "No" if none of these hold. Then complete the graph near $x = 4$ to satisfy the conditions in the final row.



a	$\lim_{x\to a^-} f(x)$	$\lim_{x\to a^+} f(x)$	$\lim_{x\to a} f(x)$	$f(a)$	Cont? RD?
-6					
-5					
-4					
-2					
-1					
0					
1					
2					
3					
4				5	RD

8. Excellent problem: Consider the piecewise function defined by $f(x) = \begin{cases} \frac{x^2-6x+8}{x-4} & \text{if } x < 4, \\ x^2 - 3x - 2 & \text{otherwise.} \end{cases}$

Carefully determine whether f is continuous at $x = 4$. Show all work.

9. (*a*) Consider the piecewise function $f(x) = \begin{cases} x^2 + 2x - 6, & \text{if } x < 3 \\ 4a - 3, & \text{if } x = 3 \\ 2x + a, & \text{if } x > 3 \end{cases}$. Is there any value of a that will make the function **continuous** at $x = 3$? Show your work.

(*b*) Is f continuous at 5? Explain carefully.