More on Continuity

Continuity on Intervals

As you might expect, a function is *continuous on an interval* if it is continuous at every point in the interval. Sounds simple, right? But what about at the endpoints when the interval is closed? Take a look at graph of $f(x)$ in Figure 10.6. What are the largest intervals on which f is continuous? We need to use our notion of one-sided continuity to deal with this question.

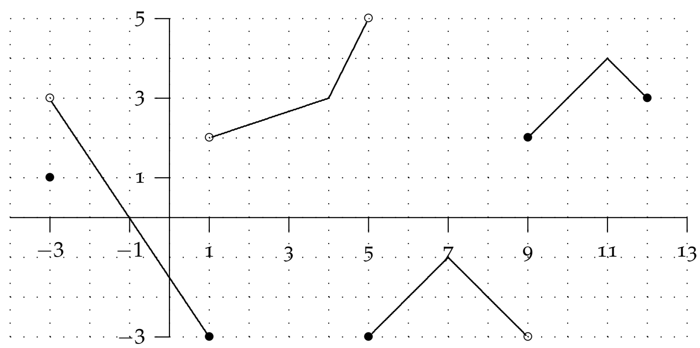

Figure 10.6: f is *left-continuous* at $x = 1$ and *right-continuous* at $x = 9$.

DEFINITION 10.1.12. (Continuity on an Interval) A function f is **continuous on an interval** I if f is continuous at all the points of I. If I contains its endpoints, then continuity at these endpoints means left-continuous for the right endpoint and right-continuous for the left endpoint.

YOU TRY IT 10.8. Use Figure 10.6 to answer these questions. Notice that f is continuous on the interval $(-3, 1]$ since it is continuous from the left at 1 but not from the right at -3. Further f is not continuous a larger interval containing this one since f is not continuous (from both sides) at either -3 or 1.

(*a*) Find the largest interval contain $x = 2$ on which f is continuous. Explain.

(*b*) Find the largest interval contain $x = 5$ on which f is continuous. Explain.

(*c*) Find the largest interval contain $x = 7$ on which f is continuous. Explain.

(*d*) Find the largest interval contain $x = 9$ on which f is continuous. Explain.

Answers to YOU TRY IT 10.8 : (a) $(1, 5)$ since f is not left-continuous at 5 nor right-continuous at 1. (b) $[5, 9)$. (c) $[5, 9)$. (d) $[9, 11]$.

EXAMPLE 10.1.13 (Intervals of Continuity). Let's do a complete analysis of the function Determine the largest intervals of continuity for

$$f(x) = \begin{cases} \frac{x^2+1}{x-1} & \text{if } x > 2 \\ -2 & \text{if } x = 2 \\ \frac{x^2-2x}{x^2-5x+6} & \text{if } x < 2 \end{cases}.$$

The piecewise function consists of two rational functions and a value at a single point. On $(2, \infty)$, $f(x) = \frac{x^2+1}{x-1}$ is continuous by Theorem 8.1.4 since the denominator is never 0. So f

is continuous at least on $(2, \infty)$. On the interval $(-\infty, 2)$, $f(x) = \frac{x^2-2x}{x^2-5x+6} = \frac{x(x-2)}{(x-2)(x-3)}$ is rational and continuous by Theorem 8.1.4 since the denominator is never 0 there. The only problem is at $x = 2$.

Questions: Is $f(x)$ left or right continuous, or just plain continuous at $x = 2$? State the intervals of continuity for f using interval notation. Does it have an RD at $x = 2$? Does f have a VA at $x = 2$? State the intervals of continuity for f using interval notation. [Make sure you justify your answers as we have done below using limits.]

Solution. Is f right-continuous at 2? Well, we know that $f(2) = -2$. And

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{x^2+1}{x-1} = \frac{5}{1} = 5 \neq -2.$$

So f is not right continuous at 2. So one of the intervals of continuity is $(2, \infty)$.

Is f left-continuous at -2?

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x^2-2x}{x^2-5x+6} = \lim_{x \to 2^-} \frac{x}{x-3} = \frac{2}{-1} = -2 = f(2).$$

So f is left-continuous at 2. Therefore, f is continuous on $(-\infty, 2]$.

However that f is not continuous at 2 since f is not continuous from **both** the left and the right.

In the end, the intervals of continuity are $(-\infty, 2,] \cup (2, \infty)$.

Is there an RD at $x = 2$? No. Notice that $\lim_{x \to 2} f(x)$ DNE because the left and right limits are different. (Remember: To have an RD, we need $\lim_{x \to 2} f(x)$ to exist and be different from $f(2)$.)

Is there a VA at $x = 2$? No. Neither the left or the right limits are infinite. (Remember: To have a VA, we need $\lim_{x \to 2^-} f(x) = \pm\infty$ or $\lim_{x \to 2^+} f(x) = \pm\infty$.)

EXAMPLE 10.1.14 (Intervals of Continuity). Let's do a complete analysis of the function

$$f(x) = \begin{cases} \frac{x+1}{x} & \text{if } x > 0 \\ x^2 - x - 1 & \text{if } x \leq 0 \end{cases}.$$

The piecewise function consists of a rational function and a polynomial. On $(0, \infty)$, $f(x) = \frac{x+1}{x}$ is continuous by Theorem 8.1.4 since the denominator is never 0. So f is continuous at least on $(0, \infty)$. On the interval $(-\infty, 0)$, $f(x) = x^2 - x - 1$ is polynomial and is continuous by Theorem 8.1.4. The only problem is a $x = 0$.

Questions: Is $f(x)$ left or right continuous, or just plain continuous at $x = 0$? State the intervals of continuity for f using interval notation. Does it have an RD at $x = 0$? Does f have a VA at $x = 0$?

Solution. Is f right-continuous at 0? Well, we know that $f(0) = 0^2 - 0 = 0$. And

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\overbrace{x+1}^{1}}{\underbrace{x}_{0^+}} = +\infty$$

So f is not right continuous at 0. So one of the intervals of continuity is $(0, \infty)$.

On the interval $(-\infty, 0)$, $f(x) = x^2 - x - 1$ is polynomial and is continuous by Theorem 8.1.4. Is f left-continuous at 0?

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 - x - 1 \stackrel{\text{Poly}}{=} -1 = f(0).$$

So f is left-continuous at 0. Therefore, f is continuous on $(-\infty, 0]$.

Note, however that f is not continuous at 0 since the two one-sided limits are different there.

In the end, the intervals of continuity are $(-\infty, 0,] \cup (0, \infty)$.

Is there an RD at $x = 0$? No. Notice that $\lim_{x \to 0} f(x)$ DNE because the left and right limits are different. (Remember: To have an RD, we need $\lim_{x \to 0} f(x)$ to exist and be different from $f(2)$.)

Is there a VA at $x = 2$? Yes, because $\lim_{x \to 0^+} f(x) = +\infty$.

EXAMPLE 10.1.15 (Intervals of Continuity). Determine the largest intervals of continuity for

$$f(x) = \begin{cases} x+2 & \text{if } x \geq 3 \\ x^2 - x - 1 & \text{if } x < 3 \end{cases}.$$

Does f have an RD at $x = 3$? VA?

Solution. The piecewise function consists of two polynomials. On $(3, \infty)$, $f(x) = x + 2$ is a continuous polynomial. Is f right-continuous at 3? Well, we know that $f(3) = 3 + 2 = 5$. And

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} x + 2 = 5$$

So f is right continuous at 3. So one of the intervals of continuity is $[3, \infty)$.

On the interval $(-\infty, 3)$, $f(x) = x^2 - x - 1$ is polynomial and is continuous by Theorem 8.1.4. Is f left-continuous at 3?

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} x^2 - x - 1 \overset{\text{Poly}}{=} 5 = f(3).$$

So f is left-continuous at 3. Therefore, f is continuous on $(-\infty, 3]$. In fact, because the two one-sided limits at 3 are equal,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = 5,$$

we have $\lim_{x \to 3} f(x) = 5 = f(3)$. So f is continuous at 3 (and everywhere else since it is a polynomial if $x \neq 3$. So $f(x)$ is actually continuous for all x, i.e., on $(-\infty, \infty)$.

Since f is continuous at $x = 3$ it does not have an RD [because $\lim_{x \to 3} f(x) = f(3) = 5$] and does not have a VA since the one-sided limits were not infinite.

Continuous Functions with Roots

So far we have not mentioned root functions or fractional power functions in our list of continuous functions. Our earlier fractional power limit property said

> Assume that m and n are positive integers and that $\frac{n}{m}$ is reduced. Then
>
> $$\lim_{x \to a} [f(x)]^{n/m} = \left[\lim_{x \to a} f(x)\right]^{n/m},$$
>
> provided that $f(x) \geq 0$ for x near a if m is even.

So if when f is actually continuous at a, then this limit may be calculated as

$$\lim_{x \to a} [f(x)]^{n/m} = \left[\lim_{x \to a} f(x)\right]^{n/m} \overset{\text{Cont}}{=} [f(a)]^{n/m}.$$

This means that $[f(x)]^{n/m}$ is actually continuous at a, *provided that* $f(x) \geq 0$ *for* x *near* a *if* m *is even*.

One problem that is sometimes encountered is that f is continuous at a with $f(a) = 0$ and f is positive on one side of a and negative on the other and m is even. Under these circumstances, the analysis we just went through does not apply, because $[f(x)]^{n/m}$ will not be defined on one side of a. In these cases, we often find that $[f(x)]^{n/m}$ is either right- or left-continuous at a. The following theorem summarizes these ideas.

THEOREM 10.1.16 (Roots and Continuity). Assume that m and n are positive integers with no common factors.

(a) If m is odd, then $[f(x)]^{n/m}$ is continuous at all points where f is continuous.

(b) If m is even, then $[f(x)]^{n/m}$ is continuous at all points a where f is continuous and $f(a) > 0$. (Note: When m is even, f may be left- or right-continuous at any points a where f is continuous and $f(a) = 0$. These points should be considered separately.)

The following example illustrates the idea.

EXAMPLE 10.1.17 (Continuity and Roots). For which values of x is $f(x) = \sqrt{1-x^2}$ continuous

Solution. You should recognize this function as the equation of the upper unit semi-circle. In any event, $f(x)$ is only defined where $1 - x^2 \geq 0$, i.e., where $1 \geq x^2$. So the domain of f is $-1 \leq x \leq 1$. Now we know that $1 - x^2$ is continuous since it is a polynomial. And by the root property for limits,

$$\lim_{x\to a} \sqrt{1-x^2} \overset{\text{Root}}{=} \sqrt{\lim_{x\to a} 1-x^2} \overset{\text{Poly}}{=} \sqrt{1-a^2} = f(a)$$

whenever $1 - a^2 > 0$. So f is continuous *at least* on the open interval $(-1, 1)$. But what about the endpoints?

At $x = 1$, f is only defined on the left side of 1. Calculate the left-hand limit, making use of the one-sided limit property for fractional powers.

$$\lim_{x\to 1^-} \sqrt{1-x^2} \overset{\text{Frac Pow}}{=} \sqrt{\lim_{x\to 1^-} 1-x^2} \overset{\text{Poly}}{=} \sqrt{0} = 0.$$

Since $f(1) = 0$ also, then f is left-continuous at $x = 1$.

Similarly at $x = -1$, f is only defined on the right side of -1. This time

$$\lim_{x\to -1^+} \sqrt{1-x^2} \overset{\text{Frac Pow}}{=} \sqrt{\lim_{x\to -1^+} 1-x^2} \overset{\text{Poly}}{=} \sqrt{0} = 0.$$

Since $f(-1) = 0$, then f is left-continuous at $x = 1$. In total, then f is continuous on the closed interval $[-1, 1]$.

EXAMPLE 10.1.18 (Continuity and Roots). Determine where $f(x) = \sqrt[3]{x^4 - x + 1}$ is continuous.

Solution. Since $x^4 - x + 1$ is a polynomial, it is continuous everywhere. Since the root is odd, we can apply part (a) of Theorem 10.1.16 and conclude that $f(x) = \sqrt[3]{x^4 - x + 1}$ is continuous for all x.

Continuity Properties

Basic Continuity Properties

One of the most important facts about continuity is that it is preserved under the standard mathematical operations on functions.

THEOREM 10.1.19. (Some Types of Continuous Functions) Assume that f and g are both continuous at $x = a$ and that c is a constant. Then the following new functions are continuous at $x = a$.

(*a*) The sum and differences: $f(x) + g(x)$ and $f(x) - g(x)$

(*b*) Constant multiples: $cf(x)$

(*c*) Products: $f(x)g(x)$

(*d*) Quotients: $\dfrac{f(x)}{g(x)}$, provided $g(a) \neq 0$

(*e*) Powers: $[f(x)]^n$, where n is a positive integer.

Assume that m and n are positive integers with no common factors.

(*f*) If m is odd, then $[f(x)]^{n/m}$

(*g*) If m is even, then $[f(x)]^{n/m}$ is continuous at all points a where $f(a) > 0$.

Proof. The proofs of all of these follow from the corresponding basic limit properties. Let's prove (*c*). Since f and g are continuous, by Definition 8.1.1 this means $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$. But then by the product limit property,

$$\lim_{x\to a}[f(x)g(x)] \overset{\text{Product}}{=} \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) \overset{f,\, g \text{ cont}}{=} f(a)g(a)$$

so fg is continuous at a. The other parts can be proven in a similar fashion. □

YOU TRY IT 10.9. Try proving part (*e*). Which limit property must you use?

YOU TRY IT 10.10 (Think it through). Earlier we the basic limit properties we were able to show earlier that polynomials were continuous. Instead we could have waited until now to show polynomials are continuous using the continuity properties in Theorem 10.1.19. We know that $f(x) = x$ is a continuous function. Do you see how you prove that x^2 is continuous everywhere directly from Theorem 10.1.19? What parts of Theorem 10.1.19 would you need to use to prove that $2x^2 + 3x$ is continuous everywhere? How about a general polynomial $p(x) = a_n x^n + \cdots a_1 x + a_0$? Once you know that polynomials are continuous, which property in Theorem 10.1.19 tells you that rational functions are continuous on their domains?

More Continuous Functions

We have seen that polynomials and rational functions are continuous for all points in their domains. If you look at your calculator, you will see keys for another handful of functions that we have not yet discussed: various logs and exponentials. Along with the trig functions these are known as 'transcendental functions' because the transcend the ordinary algebraic operations of addition, subtraction, multiplication, and division, powers, and roots. There are a couple of reasons these transcendental functions appear on most calculators. First they are useful(!), and second they are 'nice.' In particular they are continuous. Specifically

Note: When we use the term 'log' in this course, we will almost always mean the natural logarithm. OK, let's take a moment to review some basic log properties.

1. $\ln x^r = r \ln x$.
2. $\ln(xy) = \ln x + \ln y$.
3. $\ln \frac{x}{y} = \ln x - \ln y$.
4. $\ln e^x = x$ and $e^{\ln x} = x$ (inverse functions).

THEOREM 10.1.20 (A list of Continuous Functions). The standard trig functions are continuous at all points in their domains. Specifically

1. $\lim_{x\to a} \sin x = \sin a$ and $\lim_{x\to a} \cos x = \cos a$ for all real numbers a.
2. $\lim_{x\to a} \sec x = \sec a$ and $\lim_{x\to a} \tan x = \tan a$ for all real numbers $a \neq, \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$.
3. $\lim_{x\to a} \csc x = \csc a$ and $\lim_{x\to a} \cot x = \cot a$ for all real numbers $a \neq 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$.
   The standard log and exponential functions are continuous at all points in their domains. Specifically
4. $\lim_{x\to a} b^x = b^a$ for any real number $b > 0$ and for all real numbers a.
5. In particular, $\lim_{x\to a} e^x = e^a$ for all real numbers a.
6. $\lim_{x\to a} \ln x = \ln a$ for all real numbers $a > 0$. Similarly, $\lim_{x\to a} \log_b x = \log_b a$.
   The inverse trig functions are also continuous on their domains. We will have more to say about them a little later in the term.

EXAMPLE 10.1.21. Use continuity to evaluate these limits.

(*a*) Determine $\lim_{x\to\pi} x^2 \sin x$.

SOLUTION. We know x^2 is a polynomial so it is continuous and the theorem above tells us $\sin x$ is continuous. So their product is continuous. Thus

$$\lim_{x\to\pi} x^2 \sin x \overset{\text{Product}}{=} \lim_{x\to\pi} x^2 \lim_{x\to\pi} \sin x \overset{\text{Poly, Trig Cont}}{=} \pi^2 \sin\pi = \pi^2 \cdot 0 = 0.$$

(*b*) Determine $\lim_{t\to 0} \frac{t^2+4}{\cos t}$.

SOLUTION. This time we have a quotient of continuous functions (polynomial, trig) which is again continuous. So

$$\lim_{t\to 0} \frac{t^2+4}{\cos t} \overset{\text{Quotient}}{=} \frac{\lim_{t\to 0} t^2+4}{\lim_{t\to 0} \cos t} \overset{\text{Poly, Trig Cont}}{=} \frac{4}{\cos 0} = \frac{4}{1} = 4.$$

(*c*) Determine $\lim_{x\to 1} 2e^x + \ln x$.

SOLUTION. This time we have a sum of continuous functions (exponential, log) which is again continuous. So

$$\lim_{x\to 1} 2e^x + \ln x \overset{\text{Sum, Const Mult}}{=} 2\lim_{x\to 1} e^x + \lim_{x\to 1} \ln x \overset{\text{Exp, Log Cont}}{=} 2 \cdot e^1 + \ln 1 = 2 \cdot e + 0 = 2e.$$

(*d*) Determine $\lim_{x\to 0} \frac{1-\cos x}{1-\sqrt{\cos x}}$.

SOLUTION. Notice that this is a '0 over 0' limit, so we can just use continuity. We first need to use conjugates, and then apply the continuity properties.

$$\lim_{x\to 0} \frac{1-\cos x}{1-\sqrt{\cos x}} = \lim_{x\to 0} \frac{1-\cos x}{1-\sqrt{\cos x}} \cdot \frac{1+\sqrt{\cos x}}{1+\sqrt{\cos x}} = \lim_{x\to 0} \frac{(1-\cos x)(1+\sqrt{\cos x})}{1-\cos x} =$$

$$\lim_{x\to 0} 1+\sqrt{\cos x} \overset{\text{Root of Trig; Cont}}{=} 1+\sqrt{\cos 0} = 1+\sqrt{1} = 2.$$

(*e*) Determine $\lim_{x\to 1} \cos(x^2-2x+1)$. This time we are stuck, This is a composite function and we have no continuity property for this, yet,