Continuity Properties

Basic Continuity Properties

One of the most important facts about continuity is that it is preserved under the standard mathematical operations on functions.

THEOREM 10.1.19. (Some Types of Continuous Functions) Assume that f and g are both continuous at $x = a$ and that c is a constant. Then the following new functions are continuous at $x = a$.

(*a*) The sum and differences: $f(x) + g(x)$ and $f(x) - g(x)$

(*b*) Constant multiples: $cf(x)$

(*c*) Products: $f(x)g(x)$

(*d*) Quotients: $\dfrac{f(x)}{g(x)}$, provided $g(a) \neq 0$

(*e*) Powers: $[f(x)]^n$, where n is a positive integer.

Assume that m and n are positive integers with no common factors.

(*f*) If m is odd, then $[f(x)]^{n/m}$

(*g*) If m is even, then $[f(x)]^{n/m}$ is continuous at all points a where $f(a) > 0$.

Proof. The proofs of all of these follow from the corresponding basic limit properties. Let's prove (*c*). Since f and g are continuous, by Definition 8.1.1 this means $\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$. But then by the product limit property,

$$\lim_{x \to a}[f(x)g(x)] \stackrel{\text{Product}}{=} \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) \stackrel{f,\, g \text{ cont}}{=} f(a)g(a)$$

so fg is continuous at a. The other parts can be proven in a similar fashion. $\square$

YOU TRY IT 10.9. Try proving part (*e*). Which limit property must you use?

YOU TRY IT 10.10 (Think it through). Earlier we the basic limit properties we were able to show earlier that polynomials were continuous. Instead we could have waited until now to show polynomials are continuous using the continuity properties in Theorem 10.1.19. We know that $f(x) = x$ is a continuous function. Do you see how you prove that x^2 is continuous everywhere directly from Theorem 10.1.19? What parts of Theorem 10.1.19 would you need to use to prove that $2x^2 + 3x$ is continuous everywhere? How about a general polynomial $p(x) = a_n x^n + \cdots a_1 x + a_0$? Once you know that polynomials are continuous, which property in Theorem 10.1.19 tells you that rational functions are continuous on their domains?

More Continuous Functions

We have seen that polynomials and rational functions are continuous for all points in their domains. If you look at your calculator, you will see keys for another handful of functions that we have not yet discussed: various logs and exponentials. Along with the trig functions these are known as 'transcendental functions' because the transcend the ordinary algebraic operations of addition, subtraction, multiplication, and division, powers, and roots. There are a couple of reasons these transcendental functions appear on most calculators. First they are useful(!), and second they are 'nice.' In particular they are continuous. Specifically

Note: When we use the term 'log' in this course, we will almost always mean the natural logarithm. OK, let's take a moment to review some basic log properties.

1. $\ln x^r = r \ln x$.
2. $\ln(xy) = \ln x + \ln y$.
3. $\ln \frac{x}{y} = \ln x - \ln y$.
4. $\ln e^x = x$ and $e^{\ln x} = x$ (inverse functions).

THEOREM 10.1.20 (A list of Continuous Functions). The standard trig functions are continuous at all points in their domains. Specifically

1. $\lim_{x\to a} \sin x = \sin a$ and $\lim_{x\to a} \cos x = \cos a$ for all real numbers a.
2. $\lim_{x\to a} \sec x = \sec a$ and $\lim_{x\to a} \tan x = \tan a$ for all real numbers $a \neq, \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$.
3. $\lim_{x\to a} \csc x = \csc a$ and $\lim_{x\to a} \cot x = \cot a$ for all real numbers $a \neq 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$.
   The standard log and exponential functions are continuous at all points in their domains. Specifically
4. $\lim_{x\to a} b^x = b^a$ for any real number $b > 0$ and for all real numbers a.
5. In particular, $\lim_{x\to a} e^x = e^a$ for all real numbers a.
6. $\lim_{x\to a} \ln x = \ln a$ for all real numbers $a > 0$. Similarly, $\lim_{x\to a} \log_b x = \log_b a$.
   The inverse trig functions are also continuous on their domains. We will have more to say about them a little later in the term.

EXAMPLE 10.1.21. Use continuity to evaluate these limits.

(*a*) Determine $\lim_{x\to\pi} x^2 \sin x$.

SOLUTION. We know x^2 is a polynomial so it is continuous and the theorem above tells us $\sin x$ is continuous. So their product is continuous. Thus

$$\lim_{x\to\pi} x^2 \sin x \overset{\text{Product}}{=} \lim_{x\to\pi} x^2 \lim_{x\to\pi} \sin x \overset{\text{Poly, Trig Cont}}{=} \pi^2 \sin\pi = \pi^2 \cdot 0 = 0.$$

(*b*) Determine $\lim_{t\to 0} \dfrac{t^2+4}{\cos t}$.

SOLUTION. This time we have a quotient of continuous functions (polynomial, trig) which is again continuous. So

$$\lim_{t\to 0} \frac{t^2+4}{\cos t} \overset{\text{Quotient}}{=} \frac{\lim_{t\to 0} t^2+4}{\lim_{t\to 0} \cos t} \overset{\text{Poly, Trig Cont}}{=} \frac{4}{\cos 0} = \frac{4}{1} = 4.$$

(*c*) Determine $\lim_{x\to 1} 2e^x + \ln x$.

SOLUTION. This time we have a sum of continuous functions (exponential, log) which is again continuous. So

$$\lim_{x\to 1} 2e^x + \ln x \overset{\text{Sum, Const Mult}}{=} 2\lim_{x\to 1} e^x + \lim_{x\to 1} \ln x \overset{\text{Exp, Log Cont}}{=} 2 \cdot e^1 + \ln 1 = 2 \cdot e + 0 = 2e.$$

(*d*) Determine $\lim_{x\to 0} \dfrac{1-\cos x}{1-\sqrt{\cos x}}$.

SOLUTION. Notice that this is a '0 over 0' limit, so we can just use continuity. We first need to use conjugates, and then apply the continuity properties.

$$\lim_{x\to 0} \frac{1-\cos x}{1-\sqrt{\cos x}} = \lim_{x\to 0} \frac{1-\cos x}{1-\sqrt{\cos x}} \cdot \frac{1+\sqrt{\cos x}}{1+\sqrt{\cos x}} = \lim_{x\to 0} \frac{(1-\cos x)(1+\sqrt{\cos x})}{1-\cos x} =$$
$$\lim_{x\to 0} 1+\sqrt{\cos x} \overset{\text{Root of Trig; Cont}}{=} 1+\sqrt{\cos 0} = 1+\sqrt{1} = 2.$$

(*e*) Determine $\lim_{x\to 1} \cos(x^2-2x+1)$. This time we are stuck, This is a composite function and we have no continuity property for this, yet,

EXAMPLE 10.1.22. Again, these new limit properties may be combined with the previous limit properties to simplify the calculation of complicated-looking limits.

(*a*) $\lim\limits_{x\to 2} x^3 3^x \stackrel{\text{Product}}{=} (\lim\limits_{x\to 2} x^3)\cdot(\lim\limits_{x\to 2} 3^x) \stackrel{\text{Poly, Exp Cont}}{=} 2^3\cdot 3^2 = 72.$

(*b*) $\lim\limits_{x\to e} \ln x^5 \stackrel{\text{Log Property}}{=} \lim\limits_{x\to e} 5\ln x \stackrel{\text{Log Cont}}{=} 5\ln e = 5\cdot 1 = 5$. Here we used the log property that $\ln a^r = r\ln a$.

Take-home Message: Most of the familiar functions are continuous at all points in their domains. This includes: polynomials, rational functions, roots, trig, log, and exponential functions. Consequently, using Theorem 10.1.19, their sums, products, quotients, and powers are continuous as well.

Review of Composition

So far we have not mentioned the operation of composition. This is the most complex of the basic function operations, nonetheless it preserves continuity under the appropriate circumstances. Functions, like numbers, may be combined using sums, differences, products, or quotients as we have seen in our limit properties. The process called *composition* also produces new functions and this process is unique to functions and is not possible to carry out on numbers. Composition of functions will play an important role in our development of the calculus. Recall how composition works.

DEFINITION (Composite Functions). Given two functions f and g, the **composite function** $f\circ g$ is defined by $(f\circ g)(x) = f(g(x))$. It is evaluated in two steps: $y = f(u)$ where $u = g(x)$. The domain of $f\circ g$ consists of all x in the domain of g such that $u = g(x)$ is in the domain of f.

EXAMPLE 10.1.23. Let $f(x) = 3x^2 - x$, $g(x) = x^2 - 1$, $h(x) = \sqrt{x}$, and $j(x) = \frac{1}{x}$. Simplify the following expressions. Determine their domains.

(*a*) $(f\circ g)(x)$ (*b*) $(g\circ f)(x)$ (*c*) $(h\circ f)(x)$

(*d*) $(g\circ g)(x)$ (*e*) $(j\circ j)(x)$ (*f*) $(f\circ j)(x)$

Solution. We use the definition of composite function.

(*a*) $(f\circ g)(x) = f(g(x)) = f(x^2-1) = 3(x^2-1)^2 - (x^2-1) = 3x^4 - 6x^2 + 3 - x^2 + 1 = 3x^4 - 7x^2 + 4$. The domain is all x since both functions are polynomials.

(*b*) $(g\circ f)(x) = g(f(x)) = g(3x^2 - x) = (3x^2-x)^2 - 1 = 9x^4 - 6x^3 + x^2 - 1$. The domain is all x as above. This illustrates the fact that $(f\circ g)(x)$ and $(g\circ f)(x)$ are quite different functions.

(*c*) $(h\circ f)(x) = h(f(x)) = h(3x^2 - x) = \sqrt{3x^2 - x}$. This time the domain of f is all x but the domain of $h\circ f$ requires $f(x) = 3x^2 - x = x(3x-1)$ to be at least 0. So we need $x \geq \frac{1}{3}$ or $x \leq 0$.

(*d*) $(g\circ g)(x) = g(g(x)) = g(x^2-1) = (x^2-1)^2 - 1 = x^4 - 2x^2$. The domain is all x since g is a polynomial.

(*e*) $(j\circ j)(x) = j(j(x)) = j(\frac{1}{x}) = \frac{1}{\frac{1}{x}} = x$. Careful: The domain is all $x \neq 0$ since x must be in the domain of j to start with.

(*f*) $(f\circ j)(x) = f(j(x)) = f(\frac{1}{x}) = 3(\frac{1}{x})^2 - \frac{1}{x} = \frac{3}{x^2} - \frac{1}{x}$. The domain is all $x \neq 0$.

Composition, Continuity, and Limits

THEOREM 10.1.24. (Continuity and Composition)

1. If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a, which means $\lim_{x \to a} f(g(x)) = f(g(a))$.
2. If $\lim_{x \to a} g(x) = L$ and f is continuous at L, then $\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(L))$.

In the second part of this theorem we do not assume that $\lim_{x \to a} g(x) = g(a)$. There may be an RD (hole) in the graph of g at a.

The proof of this theorem is harder than the previous results—take Math 331. Part (2) of the theorem says that we can *switch in the order of the limit and composition operations* as long as f is continuous and g has a limit at a. This is also what the first part of the theorem says. When g is continuous at a, this means that $\lim_{x \to a} g(x) = g(a)$. Part (1) says

$$\lim_{x \to a} f(g(x)) \stackrel{\text{Cont}}{=} f \overbrace{(g(a))}^{\lim_{x \to a} g(x)} = f\left(\lim_{x \to a} g(x)\right). \tag{10.1}$$

In other words, we can switch the order of the limit and composition operations.

EXAMPLE 10.1.25 (Composition). Determine $\lim_{x \to 1} (x^4 - 3)^6$.

Solution. OK, we could multiply out the polynomial to the sixth power and take the limit since it will still be a polynomial. (Good luck!) Or we can think about $(x^4 - 3)^6$ as a composition: Let $g(x) = x^4 - 3$ and $f(x) = x^6$. Then

$$(x^4 - 3)^6 = f(g(x)).$$

Since both f and g are polynomials, both are continuous everywhere, so Theorem 10.1.24 applies. So switching the order of operations

$$\lim_{x \to 1} (x^4 - 3)^6 \stackrel{\text{Theorem 10.1.24}}{=} \left(\lim_{x \to 1} (x^4 - 3)\right)^6 \stackrel{\text{Poly}}{=} (1 - 3)^6 = 64.$$

EXAMPLE 10.1.26. Again, these new limit properties may be combined with the previous limit properties to simplify the calculation of complicated-looking limits.

(*a*) $\lim_{x \to 3} \ln(x^2 + 1) \stackrel{\text{Cont Composite}}{=} \ln\left(\lim_{x \to 3} x^2 + 1\right) \stackrel{\text{Poly}}{=} \ln 10.$

(*b*) Here's an example (due to Prof. Bell) where we need to use part (2) of the theorem above. Determine $\lim_{x \to 1} \sin\left(\frac{\pi x^2 - \pi}{x - 1}\right)$. If we just substituted in, we'd get $\sin \frac{0}{0}$, which is meaningless. But we can use part (2) of Theorem 10.1.24. Here $g(x) = \frac{\pi x^2 - \pi}{x - 1}$ is the inside function. It is not continuous at $x = 1$ but

$$\lim_{x \to 1} \frac{\pi x^2 - \pi}{x - 1} = \lim_{x \to 1} \frac{\pi(x^2 - 1)}{x - 1} = \lim_{x \to 1} \frac{\pi(x - 1)(x + 1)}{x - 1} = \lim_{x \to 1} \pi(x + 1) \stackrel{\text{Poly}}{=} 2\pi,$$

so the limit exists. Now by part (2) of the theorem,

$$\lim_{x \to 1} \sin\left(\frac{\pi x^2 - \pi}{x - 1}\right) \stackrel{\sin x \text{ cont}}{=} \sin\left(\lim_{x \to 1} \frac{\pi x^2 - \pi}{x - 1}\right) = \sin(2\pi) = 0.$$

(*c*) $\lim_{t \to 1} e^{t^2 - 1} \stackrel{\text{Cont Composite}}{=} e^{\lim_{t \to 1}(t^2 - 1)} \stackrel{\text{Poly,}}{=} e^0 = 1.$

(*d*) $\lim_{x \to 1/2} \sin \frac{\pi x}{2} \stackrel{\text{Cont Composite}}{=} \sin\left(\lim_{x \to 1/2} \frac{\pi x}{2}\right) \stackrel{\text{Poly}}{=} \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$

(*e*) $\lim_{x \to \pi/2} \sin[\cos(x)] \stackrel{\text{Cont Composite}}{=} \sin\left(\lim_{x \to \pi/2} \cos x\right) \stackrel{\text{Trig}}{=} \sin(\cos(\pi/2)) = \sin 0 = 0.$

(*f*) $\lim_{x \to 1} \cos(x^2 - 2x + 1) \stackrel{\text{Cont Composite}}{=} \cos\left(\lim_{x \to 0} x^2 - 2x + 1\right) \stackrel{\text{Trig}}{=} \cos(0) = 1.$

The Intermediate Value Theorem

Suppose that we want to solve an equation of the form $f(x) = L$. Often we want to find the root(s) of a function, so we want to solve $f(x) = 0$. Sometimes it is hard to find an exact solution, but nonetheless under certain circumstances we can tell that such a solution must exist (even if we don't know exactly what it is). If f is continuous, the Intermediate Value Theorem below can be helpful in many situations.

THEOREM 10.1.27 (IVT: The Intermediate Value Theorem)**.** Assume that

- f is continuous on the closed interval $[a, b]$ and
- L is a number between $f(a)$ and $f(b)$.

Then there is at least one number c in (a, b) so that $f(c) = L$.

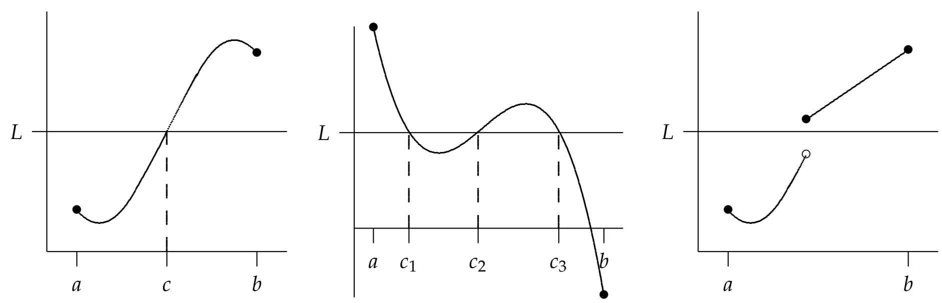

Figure 10.7: Left and middle: If f is continuous on the closed interval $[a, b]$ with L between $f(a)$ and $f(b)$, then there is at least one point c between a and b such that $f(b) = L$. Right: When f is not continuous, such a point c need not exist.

Though this theorem may seem 'obvious', the proof is surprisingly difficult and is covered in Math 331. Nonetheless, you can see how the hypothesis that f is continuous is critical. See the right-hand graph in Figure 10.7. When f is not continuous, the curve can 'jump over' the value L so that there is no point c in (a, b) with $f(c) = l$.

EXAMPLE 10.1.28. Show that $f(x) = 6x^4 + 4x^3 - 2x^2 - x - 3$ has a root in the interval $[-1, 1]$.

Solution. We want to solve $f(x) = 0$. Well, we probably are not going to factor this polynomial. Can we apply the IVT, Theorem 10.1.27? We need to check the hypotheses of the IVT.

- Is $f(x)$ continuous on the interval $[-1, 1]$? Yes! Because f is a polynomial.
- Is L between $f(a)$ and $f(b)$? Here a and b are the interval endpoints, namely -1 and 1. Ok, so $f(a) = f(-1) = 6 - 4 - 2 + 1 - 3 = -2$ and $f(1) = 6 + 4 - 2 - 1 - 3 = 4$. But what is L? In this problem we want to find where $f(x) = 0$, so $L = 0$. So, yes, $L = 0$ is between $f(-1) = -2$ and $f(1) = 4$.

So we can apply the IVT and say that there is some number c in $(-1, 1)$ so that $f(c) = L = 0$. We don't know the value of c, just that it exists. Neat!! (This is why theorems like the IVT are sometimes called **existence theorems**.)

EXAMPLE 10.1.29. Prove that $f(x) = x - \cos x = 0$ at some point in the interval $[0, 1]$.

Solution. We can't factor this function. Can we apply the IVT, Theorem 10.1.27? Check the hypotheses of the IVT.

- Is $f(x)$ continuous on the interval $[0, 1]$? Yes! Because f is the difference of continuous functions, x and $\cos x$.

- Is L between $f(a)$ and $f(b)$, where $a = 0$ and $b = 1$ are the endpoints of the interval? Well, $f(0) = 0 - \cos 0 = -1$ and $f(1) = 1 - \cos 1 = ?$ We could take out a calculator, to find the approximate value, but we can also just think about it and say that since $\cos x < 1$ when x is not a multiple of 2π, then $f(1) = 1 - \cos 0 > 0$. So $L = 0$ is between $f(0)$ and $f(1)$.

So we can apply the IVT and say that there is some number c in $(0, 1)$ so that $f(c) = L = 0$. Again, we don't know the value of c, just that it exists.

EXAMPLE 10.1.30. Show that there is some x in the interval $[0, 2]$ such that $x^2 + \cos(\pi x) = 4$.

Solution. Apply the IVT, Theorem 10.1.27. Check the hypotheses.

- Is $f(x)$ continuous on the interval $[0, 2]$? First, $\cos(\pi x)$ is a composite of trig function and a polynomial, πx. Next, x^2 is a polynomial and so is continuous. So f is the sum of continuous functions and so it is continuous on the interval $[0, 2]$.
- Is $L = 4$ between $f(0)$ and $f(2)$? Well, $f(0) = 0 + \cos 0 = 1$ and $f(2) = 4 + \cos 2\pi = 4 + 1 = 5$. So $L = 4$ is between $f(0) = 1$ and $f(2) = 5$.

So we can apply the IVT and say that there is some number c in $(0, 2)$ so that $f(c) = L = 4$. Again, we don't know the value of c, just that it exists.

YOU TRY IT 10.11. Show that $f(x) = x^3 - 4x + \cos(\pi x) = 0$ in the interval $[0, 1]$.

ANSWER TO YOU TRY IT 10.12. We want to solve $f(x) = 0$. Again we can't factor this function. But we can apply the IVT, Theorem 10.1.27. Check the hypotheses of the IVT.

- Is $f(x)$ continuous on the interval? First, $\cos(\pi x)$ is a composite of trig function and a polynomial, πx. Next, $x^3 - 4x$ is a polynomial and so is continuous. So f is the sum of continuous functions and so it is continuous on the interval $[0, 1]$.
- At the endpoints $f(0) = 0 - 0 + 1 =$ and $f(1) = 1 - 4 - 1 = -4$. Notice that 0 is between $f(0) = 1$ and $f(1) = -4$.

So by the IVT, there is some number c in $(0, 1)$ so that $f(c) = 0$. Again, we don't know the value of c, but it does exist!

YOU TRY IT 10.12 (Hand in for extra credit). . Show that $p(x) = x^4 - x^3 + x^2 + x - 1$ has at least two roots in $[-1, 1]$. Big hint: Split $[-1, 1]$ in half into two smaller closed intervals. Show that p has a root in each of the two smaller intervals.

EXAMPLE 10.1.31. Your parents invest \$20,000 in a savings account for you when you are 8-years-old. They want it to be worth \$50,000 ten years later when when you start HWS. If the account has an annual interest rate r, with monthly compounding, Then the amount in the account after 10 years (120 months) is

$$A(r) = 20,000\left(1 + \frac{r}{12}\right)^{120}.$$

Use the IVT to show that there is a value r in $(0, 0.10)$—i.e., an interest rate between 0 and 10% so that $A(r) = 50,000$.

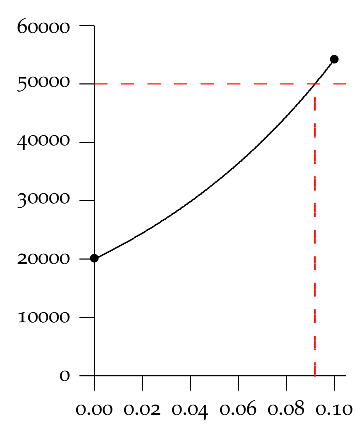

Figure 10.8: The function $A(r)$ on the interval $[0, 0.10]$. By the IVT, it must meet the horizontal line representing \$50,000 at some point in the interval.

Solution. The function $A(r) = 20,000\left(1 + \frac{r}{12}\right)^{120}$ is a composition of continuous functions (or a polynomial of degree 120 if you multiply it out!) At any rate, it is as continuous function for all values of r. So we can apply the IVT, Theorem 10.1.27 on $[0, 0.10]$. $A(0) = 20,000$—the bank is paying no interest. $A(0.10) = 54140.83$. So we have

$$A(0) < 50000 < A(0.10)$$

so by the IVT, there is a value of r in $(0, 0.10)$ so that $A(r) = 50,000$.