Basic Derivative Properties

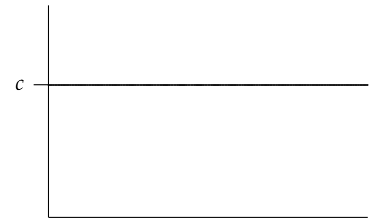


Figure 12.1: The slope of a horizontal line $y = c$ is 0 at each point, so the derivative is 0 at any point.

Let's start this section by reminding ourselves that the derivative is the slope of a function. What is the slope of a constant function?

FACT 12.1. Let $f(x) = c$, where c is a constant. Then $f'(x) = 0$.

Proof. If $f(x) = c$, then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} \frac{0}{h} = 0.$$

□

Geometrically this result is obvious. The graph of the constant function $f(x) = c$ is a horizontal line with height c. The geometrically the derivative of f is *slope* of the graph of the function. Because the graph is a horizontal line, the slope of the function is 0 at each point, exactly as we found with the limit process.

Differentiability and Continuity

If we think of continuous function as meaning the the graph is 'unbroken'—there are no gaps or holes, then a differentiable function is one that it s 'smooth'—it has no corners, let alone gaps or holes. Intuitively then, it would seem that a differentiable function must be continuous. This is, indeed, the case.

THEOREM 12.0.1. [Differentiable implies Continuous] If f is differentiable at $x = a$, then f is continuous at $x = a$.

Proof. Because f is differentiable at $x = a$, we know two things. First, by definition of the derivative,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{exists.} \tag{12.1}$$

Second, whenever $x \neq a$, then

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a} \cdot (x - a),$$

or, whenever $x \neq a$, then

$$f(x) = \frac{f(x) - f(a)}{x - a} \cdot (x - a) + f(a). \tag{12.2}$$

Now take the limit in equation (12.2) using the definition of the derivative in equation (12.1):

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\cdot(x-a)+f(a)\right)\\ &= f'(a)\cdot 0+f(a)\\ &= f(a).\end{aligned}$$

Since $\lim_{x\to a} f(x) = f(a)$, by definition f is continuous at $x = a$. $\square$

Sometimes saying the same thing in a different way can be more helpful. Notice that by the theorem above we can say: If f is not continuous, then f cannot be differentiable, because if it were differentiable Theorem 12.0.1 would say that f had to be continuous. In other words,

THEOREM 12.0.2. [Not Continuous implies Not Differentiable] If f is not continuous at $x = a$, then f is not differentiable at $x = a$.

A word of caution—Other ways to fail to be differentiable. Be careful. Neither version of the theorem tells us what happens if we start with a continuous function. In particular, if f is continuous at $x = a$, we cannot say whether f is differentiable.

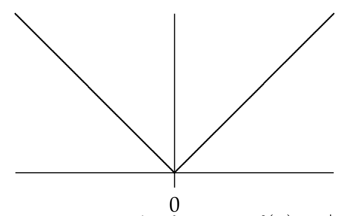


Figure 12.2: The function $f(x) = |x|$ is defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0\\ -x, & \text{if } x < 0\end{cases}$$

is continuous at $x = 0$, but is not differentiable there because there is a 'corner'. The slope as x approaches 0 from the left is -1 and from the right is $+1$.

EXAMPLE 12.0.3 (Example of a continuous function that is not differentiable.). Show that $f(x) = |x|$ is continuous at $x = 0$ but is not differentiable there.

SOLUTION. First we need to show that $|x|$ is continuous at 0. This means showing that $\lim_{x\to 0} |x| = |0|$. This is obvious from the graph (see Figure 12.2), but let's check it .

Because $|x|$ is a piecewise function, we need to check the left and right limits as x approaches 0. From the left

$$\lim_{x\to 0^-} |x| \overset{x\leq 0}{=} \lim_{x\to 0^-} -x \overset{\text{linear}}{=} 0.$$

From the right,

$$\lim_{x\to 0^+} |x| \overset{x\geq 0}{=} \lim_{x\to 0^+} x \overset{\text{linear}}{=} 0.$$

Since the left and right limits both are 0, it follows that $\lim_{x\to 0} |x| = 0$. Since $f(0) = |0| = 0$, we see that $f(x)$ is continuous at $x = 0$.

To show that f is not differentiable at 0, we use the definition of the derivative of f at a point,

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}.$$

Again, because of the piecewise nature of the $|x|$, we must check the one-sided limits. So From the left

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-}\frac{|x|-|0|}{x-0} \overset{x\leq 0}{=} \lim_{x\to 0^-}\frac{-x}{x} = -1.$$

From the right,

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}\frac{|x|-|0|}{x-0} \overset{x\geq 0}{=} \lim_{x\to 0^+}\frac{x}{x} = 1.$$

Since the left and right limits are different, it follows that $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$ does not exist. In other words, the function is not differentiable at $x = 0$.

Here are four examples of functions that are continuous at $x = a$ but which are not differentiable.

(a)

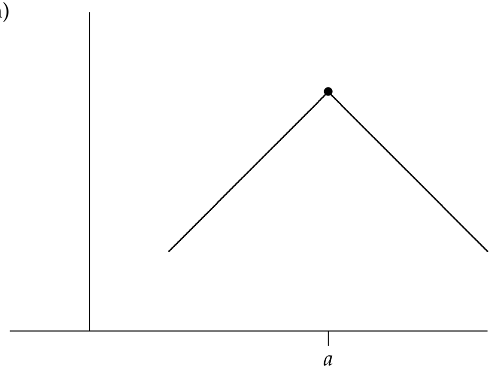
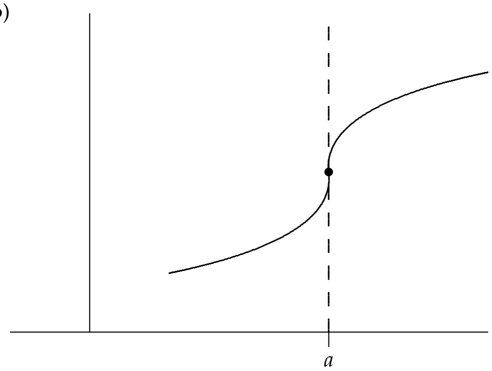


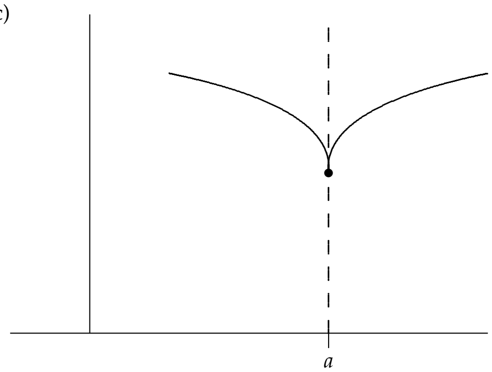
Figure 12.3: (a) The graph is continuous (unbroken) but has a *corner*, so there are two different slopes at $x = a$ depending on whether we approach a from the left or the right. So f is not differentiable at a, that is, $f'(a)$ does not exist.

(b)



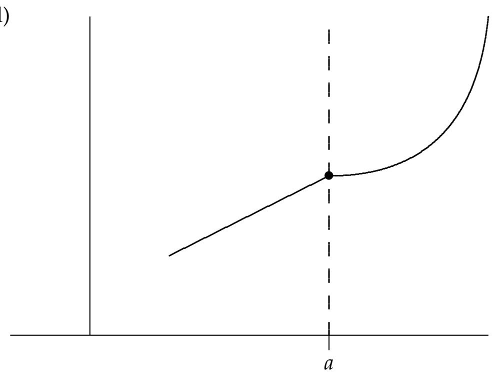
(b) The graph is continuous (unbroken) but has a a *vertical tangent* at $x = a$ (infinite slope). So f is not differentiable at a, that is, $f'(a)$ does not exist.

(c)



(c) The graph is continuous (unbroken) but has a a *cusp* at $x = a$. So f is not differentiable at a, that is, $f'(a)$ does not exist.

(d)



(d) The graph is continuous (unbroken) but has a a *corner* at $x = a$. So there are two different slopes at $x = a$ depending on whether we approach a from the left or the right. So f is not differentiable at a, that is, $f'(a)$ does not exist.