## Basic Derivative Properties

Let's start this section by reminding ourselves that the derivative is the slope of a function. What is the slope of a constant function?

FACT 12.1. Let $f(x)=c$, where $c$ is a constant. Then $f^{\prime}(x)=0$.

Proof. If $f(x)=c$, then

$$
f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}=\lim _{h \rightarrow 0} \frac{c-c}{h}=\lim _{h \rightarrow 0} \frac{0}{h}=0
$$

Geometrically this result is obvious. The graph of the constant function $f(x)=c$ is a horizontal line with height $c$. The geometrically the derivative of $f$ is slope of the graph of the function. Because the graph is a horizontal line, the slope of the function is 0 at each point, exactly as we found with the limit process.

## Differentiability and Continuity

If we think of continuous function as meaning the the graph is 'unbroken'-there are no gaps or holes, then a differentiable function is one that it s 'smooth'—it has no corners, let alone gaps or holes. Intuitively then, it would seem that a differentiable function must be continuous. This is, indeed, the case.

THEOREM 12.0.1. [Differentiable implies Continuous] If $f$ is differentiable at $x=a$, then $f$ is continuous at $x=a$.

Proof. Because $f$ is differentiable at $x=a$, we know two things. First, by definition of the derivative,

$$
\begin{equation*}
f^{\prime}(a)=\lim _{x \rightarrow a} \frac{f(x)-f(a)}{x-a} \quad \text { exists. } \tag{12.1}
\end{equation*}
$$

Second, whenever $x \neq a$, then

$$
f(x)-f(a)=\frac{f(x)-f(a)}{x-a} \cdot(x-a)
$$

or, whenever $x \neq a$, then

$$
\begin{equation*}
f(x)=\frac{f(x)-f(a)}{x-a} \cdot(x-a)+f(a) \tag{12.2}
\end{equation*}
$$

Now take the limit in equation (12.2) using the definition of the derivative in equation (12.1):

$$
\begin{aligned}
\lim _{x \rightarrow a} f(x) & =\lim _{x \rightarrow a}\left(\frac{f(x)-f(a)}{x-a} \cdot(x-a)+f(a)\right) \\
& =f^{\prime}(a) \cdot 0+f(a) \\
& =f(a) .
\end{aligned}
$$

Since $\lim _{x \rightarrow a} f(x)=f(a)$, by definition $f$ is continuous at $x=a$.
Sometimes saying the same thing in a different way can be more helpful. Notice that by the theorem above we can say: If $f$ is not continuous, then $f$ cannot be differentiable, because if it were differentiable Theorem 12.0.1 would say that $f$ had to be continuous. In other words,

THEOREM 12.0.2. [Not Continuous implies Not Differentiable] If $f$ is not continuous at $x=$ $a$, then $f$ is not differentiable at $x=a$.

A word of caution-Other ways to fail to be differentiable. Be careful. Neither version of the theorem tells us what happens if we start with a continuous function. In particular, if $f$ is continuous at $x=a$, we cannot say whether $f$ is differentiable.

EXAMPLE 12.0.3 (Example of a continuous function that is not differentiable.). Show that $f(x)=|x|$ is continuous at $x=0$ but is not differentiable there.

SOLUTION. First we need to show that $|x|$ is continuous at 0 . This means showing that $\lim _{x \rightarrow 0}|x|=|0|$. This is obvious from the graph (see Figure 12.2), but let's check it

Because $|x|$ is a piecewise function, we need to check the left and right limits as $x$ approaches 0. From the left

$$
\lim _{x \rightarrow 0^{-}}|x|^{x} \leqq 0 \lim _{x \rightarrow 0^{-}}-x \stackrel{\text { linear }}{=} 0 .
$$

From the right,

$$
\lim _{x \rightarrow 0^{+}}|x|^{x} \geqq 0 \lim _{x \rightarrow 0^{+}} x \stackrel{\text { linear }}{=} 0 .
$$

Since the left and right limits both are 0 , it follows that $\lim _{x \rightarrow 0}|x|=0$. Since $f(0)=|0|=$ 0 , we see that $f(x)$ is continuous at $x=0$.

To show that $f$ is not differentiable at 0 , we use the definition of the derivative of $f$ at a point,

$$
\lim _{x \rightarrow 0} \frac{f(x)-f(0)}{x-0}
$$

Again, because of the piecewise nature of the $|x|$, we must check the one-sided limits. So From the left

$$
\lim _{x \rightarrow 0^{-}} \frac{f(x)-f(0)}{x-0}=\lim _{x \rightarrow 0^{-}} \frac{|x|-|0|}{x-0} \stackrel{x}{ } \leqq 0 \lim _{x \rightarrow 0^{-}} \frac{-x}{x}=-1 .
$$

From the right,

$$
\lim _{x \rightarrow 0^{+}} \frac{f(x)-f(0)}{x-0}=\lim _{x \rightarrow 0^{+}} \frac{|x|-|0|}{x-0} \stackrel{x}{x} \geqq \lim _{x \rightarrow 0^{+}} \frac{x}{x}=1 .
$$

Since the left and right limits are different, it follows that $\lim _{x \rightarrow 0} \frac{f(x)-f(0)}{x-0}$ does not exist. In other words, the function is not differentiable at $x=0$.

Here are four examples of functions that are continuous at $x=a$ but which are not differentiable.
(a)

(c)


Figure 12.3: (a) The graph is continuous (unbroken) but has a corner, so there are two different slopes at $x=a$ depending on whether we approach $a$ from the left or the right. So $f$ is not differentiable at $a$, that is, $f^{\prime}(a)$ does not exist.
(b) The graph is continuous (unbroken) but has a a vertical tangent at $x=a$ (infinite slope). So $f$ is not differentiable at $a$, that is, $f^{\prime}(a)$ does not exist.
(c) The graph is continuous (unbroken) but has a a cusp at $x=a$. So $f$ is not differentiable at $a$, that is, $f^{\prime}(a)$ does not exist.
(d) The graph is continuous (unbroken) but has a corner at $x=a$. So there are two different slopes at $x=a$ depending on whether we approach $a$ from the left or the right. So $f$ is not differentiable at $a$, that is, $f^{\prime}(a)$ does not exist.

## 5-Minute Review: Noticing a Pattern

In the last few days we have calculated several derivatives. Using the calculations from our previous class, the lab ticket, and lab fill in the derivatives of each of these power functions.

| $f(x)$ | $x^{3}$ | $x^{2}$ | $x$ | $x^{-1}$ | $x^{-2}$ | $\sqrt{x}=x^{1 / 2}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $f^{\prime}(x)$ |  |  |  |  |  |  |

If you spot a pattern, if $f(x)=x^{4}$ what should $f^{\prime}(x)$ be? If $f(x)=x^{-4}$, what should $f^{\prime}(x)$ be? Bonus: If $f(x)=\frac{1}{\sqrt{x}}$, what should $f^{\prime}(x)$ be?

## Other Notation for Derivatives

The development of calculus is often attributed to two people, Isaac Newton and Gottfried Leibniz, who independently worked its foundations. Although they both were instrumental in its creation, they thought of the fundamental concepts in very different ways. It is interesting to note that Leibniz was very conscious of the importance of good notation and put a lot of thought into the symbols he used. Newton, on the other hand, wrote more for himself than anyone else. Consequently, he tended to use whatever notation he thought of on that day. This turned out to be important in later developments. Leibniz's notation was better suited to generalizing calculus to multiple variables and in addition it highlighted the operator aspect of the derivative and integral. As a result, much of the notation that is used in Calculus today is due to Leibniz.

Nonetheless, there are several ways are commonly used to indicate the derivative of a function.

NOTATION. Assume that $y=f(x)$ is a differentiable function. Then the derivative function of $f$ may be denoted by any of the following:

$$
f^{\prime}(x)=y^{\prime}=y^{\prime}(x)=\frac{d y}{d x}=\frac{d}{d x}(f(x))=\frac{d f}{d x}=D_{x}(f(x)) .
$$

The notation $\frac{d y}{d x}$ is meant to remind us of the slope formula $\frac{\Delta y}{\Delta x}$ that motivates the definition of the derivative. The notation $D_{x}(f(x))$ is especially useful because we can use it to display both the original function and its derivative quite compactly. For example, instead of writing the sentence

$$
\begin{equation*}
\text { If } f(x)=x^{2}, \text { then } f^{\prime}(x)=2 x \tag{12.3}
\end{equation*}
$$

we can write

$$
\begin{equation*}
D_{x}\left(x^{2}\right)=2 x \quad \text { or } \quad \frac{d}{d x}\left(x^{2}\right)=2 x \tag{12.4}
\end{equation*}
$$

2 This raises an important point with which beginning calculus students sometimes have difficulty: using proper notation. It is all too easy to incorrectly write

$$
\begin{equation*}
f(x)=x^{2}=2 x \tag{12.5}
\end{equation*}
$$

This confuses the original function with its derivative. Obviously the two functions $x^{2}$ and $2 x$ are not the same, but that's what the equal sign means in equation (12.5). So be sure to use the correct notation as in (12.3) or (12.4) above.

This paragraph is taken from "The History of Calculus" at https://www. math.uh.edu/~tomforde/calchistory. html

NOTATION. To reference the derivative of a function at a specific point when $x=a$, use the any of the following notation:

$$
f^{\prime}(a)=y^{\prime}(a)=\left.\frac{d y}{d x}\right|_{x=a}=\left.\frac{d f}{d x}\right|_{x=a}
$$

For example, if $f(x)=x^{2}$, we have seen that $f^{\prime}(x)=2 x$. So to indicate the slope or instantaneous rate of change in $f$ at $x=5$, we might write

$$
f^{\prime}(5)=10 \quad \text { or }\left.\quad \frac{d y}{d x}\right|_{x=5}=10
$$

Let's use our new notation in determining another derivative.
EXAMPLE 12.1.4. Determine the derivative of $f(x)=\frac{1}{\sqrt{x}}$. Note: The domain of $f(x)$ is $(0, \infty)$.
SOLUTION. Using the definition of the derivative,

$$
\begin{aligned}
\frac{d}{d x}\left[\frac{1}{\sqrt{x}}\right]=\lim _{h \rightarrow 0} \frac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h}=\lim _{h \rightarrow 0} \frac{\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x+h} \cdot \sqrt{x}}}{h} & =\lim _{h \rightarrow 0} \frac{\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x+h} \cdot \sqrt{x}}}{h} \cdot \frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}} \\
& =\lim _{h \rightarrow 0} \frac{x-(x+h)}{h \cdot \sqrt{x+h} \cdot \sqrt{x} \cdot(\sqrt{x}+\sqrt{x+h})} \\
& =\lim _{h \rightarrow 0} \frac{-h}{h \cdot \sqrt{x+h} \cdot \sqrt{x} \cdot(\sqrt{x}+\sqrt{x+h})} \\
& =\lim _{h \rightarrow 0} \frac{-1}{\sqrt{x+h} \cdot \sqrt{x} \cdot(\sqrt{x}+\sqrt{x+h})} \\
& \stackrel{\text { Cont }}{=} \frac{-1}{\sqrt{x} \cdot \sqrt{x} \cdot(\sqrt{x}+\sqrt{x})} \\
& -\frac{1}{x(2 \sqrt{x})} \\
& -\frac{1}{2 x^{3 / 2}} \\
& -\frac{1}{2} x^{-3 / 2} .
\end{aligned}
$$

In other words, $D_{x}\left(x^{-1 / 2}\right)=-\frac{1}{2} x^{-3 / 2}$. This fits the earlier pattern seen in the table on page 4. It appears that $D_{x}\left(x^{r}\right)=r x^{r-1}$. It appears that $D_{x}\left(x^{r}\right)=r x^{r-1}$.

The Derivative of $f(x)=x^{n}$
With a bit of careful algebra and the use of some basic limit properties, we can now determine a general formula for the derivative of $f(x)=x^{n}$, at least when $n$ is a positive integer. (From the pattern that we have observed, we expect the answer to be $f^{\prime}(x)=n x^{n-1}$.)

Before we begin, we need to make an observation about factoring. This is best illustrated with the following example.
EXAMPLE 12.1.5. Multiply out and simplify the product $(x-a)\left(x^{3}+a x^{2}+a^{2} x+a^{3}\right)$.
SOLUTION. We find

$$
\begin{align*}
(x-a)\left(x^{3}+a x^{2}+a^{2} x+a^{3}\right) & =x^{4}+a x^{3}+a^{2} x^{2}+a^{3} x-a x^{3}-a^{2} x^{2}-a^{3} x-a^{3} \\
& =x^{4}-a^{4} \tag{12.6}
\end{align*}
$$

Notice by dividing by $x-a$ we can express this product as a quotient

$$
x^{3}+a x^{2}+a^{2} x+a^{3}=\frac{x^{4}-a^{4}}{x-a}
$$

YOU TRY IT 12.1. Notice the cancellation of all but the first and last terms in equation (12.6) above. Simplify the product

$$
(x-a)\left(x^{4}+a x^{3}+a^{2} x^{2}+a^{3} x+a^{4}\right) .
$$

For any positive integer $n>1$, Determine the following product.

$$
(x-a)\left(x^{n-1}+a x^{n-2}+a^{2} x^{n-3} \cdots a^{n-2} x+a^{n-1}\right) .
$$

Finally rewrite each expression as equivalent.

Now we are ready to prove the general result.

THEOREM 12.1.6 (Power Rule for Derivatives). Let $n$ be a positive integer. The derivative of $f(x)=$ $x^{n}$ is $f^{\prime}(x)=n x^{n-1}$.

Proof. Let $n$ be a positive integer and let $a$ be any real number. We will determine the formula for $f^{\prime}(a)$. By definition,

$$
\begin{aligned}
& f^{\prime}(a)=m_{\mathrm{tan}}=\lim _{x \rightarrow a} \frac{f(x)-f(a)}{x-a}=\lim _{x \rightarrow a} \frac{x^{n}-a^{n}}{x-a} \\
& \text { You Try It 12.1 } \lim _{x \rightarrow a} x^{n-1}+a x^{n-2}+a^{2} x^{n-3} \cdots a^{n-2} x+a^{n-1} \\
& \stackrel{\text { Poly }}{=} \overbrace{a^{n-1}+a^{n-1}+a^{n-1}+\cdots+a^{n-1}+a^{n-1}} \\
&=n a^{n-1} .
\end{aligned}
$$

That is, $f^{\prime}(a)=n a^{n-1}$, so $f^{\prime}(x)=n x^{n-1}$. That is, for any positive integer $n$,

$$
\frac{d}{d x}\left[x^{n}\right]=n x^{n-1}
$$

Wow! This means we can calculate lot's of derivatives without having to evaluate a limit expression.

EXAMPLE 12.1.7. Determine the derivatives of $f(x)=x^{9}, g(t)=t^{99}$, and $f(s)=s^{203}$.
SOLUTION. Using Theorem 12.1.6, we know that $\frac{d}{d x}\left[x^{n}\right]=n x^{n-1}$ so $D_{x}\left(x^{9}\right)=9 x^{8}$, $D_{t}\left(t^{99}\right)=99 t^{98}$, and $D_{s}\left(s^{203}\right)=203 s^{202}$. Notice the use of proper notation, including the correct variable names.

Though we have not proven it yet, there is a more general version of the power rule that applies to any non-zero exponent that has the same form as the Power Rule for Derivatives.

THEOREM 12.1.8 (General Power Rule for Derivatives). Let $r$ be any non-zero real number. Then

$$
\frac{d}{d x}\left[x^{r}\right]=r x^{r-1}
$$

EXAMPLE 12.1.9. Determine the derivatives of $f(x)=x^{2 / 3}, g(t)=t^{9 / 2}, f(s)=s^{-203}$, and $g(w)=\frac{1}{w^{5 / 4}}$.

SOLUTION. Using Theorem 12.1.8, we know that $\frac{d}{d x}\left[x^{r}\right]=r x^{r-1}$. So
(a) $D_{x}\left(x^{2 / 3}\right)=\frac{2}{3} x^{-1 / 3}$
(b) $D_{t}\left(t^{9 / 2}\right)=\frac{9}{2} t^{7 / 2}$
(c) $D_{s}\left(s^{-203}\right)=-203 s^{204}$
(d) $D_{w}\left(\frac{1}{w^{5 / 4}}\right)=D_{w}\left(w^{-5 / 4}\right)=-\frac{5}{4} w^{-9 / 4}$

Be especially careful with negative exponents to subtract 1 from the exponent. Be sure to use proper notation, including the correct variable names.

YOU TRY IT 12.2. Determine the derivatives of $f(x)=x^{5 / 9}, g(t)=\frac{1}{t}^{19}$, and $h(w)=w^{-4 / 7}$.

## Algebraic Properties of Derivatives

As we did with limits, we will now develop some basic properties of derivatives of sums and constant multiples of differentiable functions. Precisely because derivatives are limits, we can employ the corresponding limit properties to help us develop these rules.

THEOREM 12.1.10 (Constant Multiple and Sum/Difference Properties). Assume that $f$ and $g$ are differentiable functions and that $c$ is a constant. Then
(1) $\frac{d}{d x}[c f(x)]=c f^{\prime}(x)$;
(2) $\frac{d}{d x}[f(x) \pm g(x)]=f^{\prime}(x) \pm g^{\prime}(x)$. In other words, the derivative of a sum is the sum of the derivatives.

Theorem 12.1.10 is about order of operations. For example, part (2) says that we take the sum first and then calculate the derivative or we can take the individual derivatives first and then add them. The result will be the same. Most of the time it is easier to determine the individual derivatives first and then add. But occasionally, the other order is simpler. How would you describe the order of operations in part (1)?

Theorem 12.1.10 is a very powerful tool. Before we give its proof, let's examine how to use this result.

EXAMPLE 12.1.11. Determine the derivative of $h(x)=9 x^{7}-6 x^{1 / 2}+5$, where $x>0$.
SOLUTION. Using our earlier work, we know that $x^{7}, x^{1 / 2}$ and 5 are all differentiable functions. So we may apply Theorem 12.1.10. First we split $h(x)$ into three pieces,

$$
D_{x}\left(9 x^{7}-6 x^{1 / 2}+5\right) \stackrel{\text { Theorem 12.1.10, part } 2}{=} D_{x}\left(9 x^{7}\right)-D_{x}\left(6 x^{1 / 2}\right)+D_{x}(5)
$$

then factor out the constants

$$
\stackrel{\text { Theorem 12.1.10, part } 1}{=} 9 D_{x}\left(x^{7}\right)-6 D_{x}\left(x^{1 / 2}\right)+D_{x}(5)
$$

and now use the General Power Rule and the derivative of a constant function

$$
\begin{aligned}
& \text { Theorem 12..1.8, Example } 9\left(7 x^{6}\right)-6\left(\frac{1}{2} x^{-1 / 2}\right)+0 \\
& =63 x^{6}-3 x^{-1 / 2}
\end{aligned}
$$

This is substantially simpler than actually having to use a limit process to determine the derivative.

To be able to use Theorem 12.1.10 we need to demonstrate why it is true.

Proof. To prove part 1 of Theorem 12.1.10, assume that $f$ is a differentiable function and that $c$ is a constant. Then by definition of the derivative,

$$
D_{x}(c f(x))=\lim _{h \rightarrow 0} \frac{c f(x+h)-c f(x)}{h}
$$

Using the Constant Multiple Property for Limits

$$
=c \lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}
$$

An finish by using the the fact that $f$ is differentiable

$$
=c f^{\prime}(x)
$$

To prove part 2 of Theorem 12.1.10, that $f$ and $g$ are differentiable functions .
Then by definition of the derivative,

$$
D_{x}(f(x)+g(x))=\lim _{h \rightarrow 0} \frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}
$$

which we can rearrange as

$$
=\lim _{h \rightarrow 0} \frac{[f(x+h)-f(x)]+[g(x+h)-g(x)]}{h}
$$

and then use the Sum Property of Limits

$$
=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}+\lim _{h \rightarrow 0} \frac{g(x+h)-g(x)}{h}
$$

Finish by using the the fact that both $f$ are differentiable

$$
=f^{\prime}(x)+g^{\prime}(x) .
$$

Here are a few more examples.
EXAMPLE 12.1.12. Determine the derivatives of the following functions, where they exist.
(a) $f(x)=3 x^{2}-2 x^{3}+8$
(b) $g(x)=\frac{x^{5}}{10}+\frac{10}{\sqrt[3]{x}}-2$
(c) $s(t)=(t+1)\left(t^{3}+2\right)$
(d) $f(s)=\frac{8 s^{2}+7 s+2}{s}$

SOLUTION. In each of these we make use of the General Power Theorem, the Sum/Difference Theorem, and the Constant Multiple Theorem for derivatives.
(a) For $f(x)=3 x^{2}-2 x^{3}+8$,

$$
\begin{aligned}
D_{x}\left(3 x^{2}-2 x^{3}+8\right) & \stackrel{\text { Thm }}{ } \stackrel{\text { 12.1.100 }(2)}{=} D_{x}\left(3 x^{2}\right)-D_{x}\left(2 x^{3}\right)+D_{x}(8) \\
& \operatorname{Thm~}_{12.1 .100(1)}^{=} 3 D_{x}\left(x^{2}\right)-2 D_{x}\left(x^{3}\right)+D_{x}(8) \\
& \text { Thm } 12.1 .6, \text { Const } \\
= & 3 \cdot 2 x-2 \cdot 3 x^{2}+0 \\
& =6 x-6 x^{2} .
\end{aligned}
$$

(b) For $g(x)=\frac{x^{5}}{10}+\frac{10}{\sqrt[3]{x}}-2$, rewrite in exponent form

$$
\begin{aligned}
D_{x}\left(\frac{x^{5}}{10}+10 x^{-1 / 3}-2\right) & \stackrel{\operatorname{Thm}}{ } \stackrel{12.1 .10(2)}{=} D_{x}\left(\frac{x^{5}}{10}\right)+D_{x}\left(10 x^{-1 / 3}\right)-D_{x}(2) \\
& \text { Thm 12.1.10(1) } \\
= & \frac{1}{10} D_{x}\left(x^{5}\right)+10 D_{x}\left(x^{-1 / 3}\right)-D_{x}(2) \\
& \text { Thm 12.1.6, Const } \frac{1}{10} \cdot 5 x^{4}+10\left(\frac{-1}{3} x^{-4 / 3}\right)+0 \\
& =\frac{x^{4}}{2}-\frac{10 x^{-4 / 3}}{3} .
\end{aligned}
$$

In the next two we combine some steps.
(c) For $s(t)=(t+1)\left(t^{3}+2\right)$, first multiply out

$$
\begin{aligned}
D_{t}\left((t+1)\left(t^{3}+2\right)\right) & \left.=D_{t}\left(t^{4}+t^{3}+2 t+2\right)\right) \\
& \stackrel{\operatorname{Thm} \stackrel{\text { 12.1.10(2) }}{=} D_{t}\left(t^{4}\right)+D_{t}\left(t^{3}\right)+D_{t}(2 t)+D_{t}(2)}{ } \\
& \text { Thm 12.1.10(1), } \stackrel{\text { Thm 12.1.6, Const }}{=} 4 t^{3}+3 t^{2}+2+0 \\
& =4 t^{3}+3 t^{2}+2 .
\end{aligned}
$$

(d) For the final one, first simplify by dividing through by $s$.

$$
\begin{aligned}
& \left.D_{s}\left(\frac{8 s^{2}+7 s+2}{s}\right)=D_{s}\left(8 s+7+2 s^{-1}\right)\right) \\
& \stackrel{T h m}{12.1 .10(2)} D_{t}\left(t^{4}\right)+D(8 s)+D_{s}(7)+D_{s}\left(2 s^{-1}\right) \\
& \text { Thm 12.1.10(1), Thm 12.1.6, Const } 8+0-2 s^{-2} \\
& =8-2 s^{-2} \text {. }
\end{aligned}
$$

Theorem 12.1.10 will be even more useful once we have determined (general) derivative formulas for functions other than just powers of $x$ or constants. Such additional functions include exponential and $\log$ functions, the trig functions, and the inverse trig functions. In the next section we look at derivatives of the exponential function.

