## Derivatives of Exponentials

Recall that
DEFINITION 12.1.13. An exponential function has the form $f(x)=a^{x}$, where the base is a real number $a>0$. The domain of an exponential function is the set of all real numbers $(-\infty, \infty)$.

Notice that unlike power functions of the form $x^{n}$ where the base $x$ varies and the exponent is constant, in an exponential function $a^{x}$ the base $a$ is fixed and it is the exponent varies. We have mentioned previously that exponential functions are continuous. This means that $\lim _{x \rightarrow h} a^{x}=a^{h}$.

Take a moment right now to answer the following questions.
YOU TRY IT 12.3. Fill in the table below for the values of $y=a^{x}$. Then draw the graphs.
How would you describe the relationship between corresponding pairs of graphs? How are their slopes (derivatives) related?

| $x$ | $2^{x}$ | $3^{x}$ | $1^{x}$ | $\frac{1}{2}^{x}$ | $\frac{1}{3}^{x}$ |
| ---: | ---: | ---: | ---: | ---: | ---: |
| -3 | $\frac{1}{8}$ |  |  | 8 |  |
| -2 | $\frac{1}{4}$ |  |  | 4 |  |
| -1 | $\frac{1}{2}$ |  |  | 2 |  |
| 0 | 1 |  |  | 1 |  |
| 1 | 2 |  |  | $\frac{1}{2}$ |  |
| 2 | 4 |  |  | $\frac{1}{4}$ |  |
| 3 | 8 |  |  | $\frac{1}{8}$ |  |




As calculus students, the first question we should now have regarding any function is: Is $f(x)$ differentiable. . . and if so, what is $f^{\prime}(x)$ ?

So our problem is to determine $D_{x}\left(a^{x}\right)$. Let's use the definition of the derivative
and see where this takes us. So let $f(x)=a^{x}$, where $a>0$. Then

$$
\begin{align*}
f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} & =\lim _{h \rightarrow 0} \frac{a^{x+h}-a^{x}}{h} \\
& =\lim _{h \rightarrow 0} \frac{a^{x} a^{h}-a^{x}}{h} \\
& =\lim _{h \rightarrow 0} \frac{a^{x}\left(a^{h}-1\right)}{h} \\
& =a^{x} \lim _{h \rightarrow 0} \frac{a^{h}-1}{h} . \tag{12.7}
\end{align*}
$$

Notice that the expression $\lim _{h \rightarrow 0} \frac{a^{h}-1}{h}=\lim _{h \rightarrow 0} \frac{a^{h}-a^{0}}{h}$ is really just the slope of $a^{x}$ at $x=0$. There is no additional algebra that we can perform to simplify this limit. It's value is not obvious since both the numerator and denominator are both going to 0 as $h$ goes to 0 . Let's examine a table of values to estimate the limit, if it exists.
you try it 12.4. Let $a=2$. Fill in the rest of the values in this table and then estimate the value of $\lim _{h \rightarrow 0} \frac{2^{h}-1}{h}$.

| $h$ | $\lim _{h \rightarrow 0} \frac{2^{h}-1}{h}$ | $h$ | $\lim _{h \rightarrow 0} \frac{2^{h}-1}{h}$ |
| :--- | :--- | :--- | :--- |
| 0.01 | 0.6955550 | -0.01 |  |
| 0.0001 |  | -0.0001 |  |
| 0.000001 |  | -0.000001 |  |


YOU TRY IT 12.5. Remember that in equation (12.7), we found that $D_{x}\left(a^{x}\right)=a^{x} \lim _{h \rightarrow 0} \frac{a^{h}-1}{h}$, if the limit exists. So letting $a=2$ and using (approximate) value of $\lim _{h \rightarrow 0} \frac{2^{h}-1}{h}$ that you just found above, determine the formula for $D_{x}\left(2^{x}\right)$

The exercise above shows that we will know the derivative of $a^{x}$ once we are able to determine $\lim _{h \rightarrow 0} \frac{a^{h}-1}{h}$. The following problem asks you to estimate this limit for several different values of $a$. You will need a calculator. Try this now.
YOU TRY IT 12.6. For various values of $a$, we can estimate $\lim _{h \rightarrow 0} \frac{a^{h}-1}{h}$ by using by using $h=0.000001$. In other words, using a calculator we obtain a good estimate of the limit using

$$
\lim _{h \rightarrow 0} \frac{a^{h}-1}{h} \approx \frac{a^{0.000001}-1}{0.000001} .
$$

Fill in your estimates in the table below. Use at least three decimal places

| $a$ | $\frac{1}{3}$ | $\frac{1}{2}$ | 1 | 2 | 3 | 2.5 |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $\lim _{h \rightarrow 0} \frac{a^{h}-1}{h}$ |  |  |  | 0.693 |  |  |


| 9160 | 9860․ L | E6900 | 0 | 8690- | $9860{ }^{\circ} \mathrm{I}$ | $\frac{y}{L-y^{v}} \stackrel{\substack{0 \leftarrow y \\ \text { wul }}}{ }$ | 9'zi li kul nox ol yqu |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| S.z | $\varepsilon$ | z | I | $\frac{\tau}{L}$ | $\frac{\varepsilon}{\text { L }}$ | $v$ |  |

Stop and Think：If you look at the answer to You Try It 12．6，you will see that the limits for $a=\frac{1}{2}$ and $a=2$ have the same magnitude but are oppositely signed． The result is similar for $a=\frac{1}{3}$ and $a=3$ ．This makes sense and follows from two observations that we made earlier．The first observation was made in the answer to You Try It 12.3 where we noted that $a^{x}$ and $\frac{1}{a}^{x}$ are reflections of each other in the $y$－ axis，which means that the slopes at corresponding points on these curves have the same magnitude but are oppositely signed．The second observation that the limit expression $\lim _{h \rightarrow 0} \frac{a^{h}-1}{h}$ is really just the slope of $a^{x}$ at $x=0$ ．Any point with $x=0$ lies on the $y$－axis and so is reflected to itself under reflection in the $y$－axis．So the limits when $a=\frac{1}{2}$ and $a=2$ represent the slopes of the curves $y=\frac{1}{2}^{x}$ and $y=2^{x}$ at $x=0$ ，that is，at the point $(0,1)$ that lies on every the graph of every exponential function．These slopes should have the same magnitude but different signs．

So what about the Derivative of $a^{x}$ ？Remember that in equation（12．7），we found that $D_{x}\left(a^{x}\right)=a^{x} \lim _{h \rightarrow 0} \frac{a^{h}-1}{h}$ ，if the limit exists．Using the table of limits in You Try It 12．6，this means that we have

$$
\begin{array}{|c|}
\hline D_{x}\left(a^{x}\right)=a^{x} \lim _{h \rightarrow 0} \frac{a^{h}-1}{h} \\
\hline D_{x}\left(2^{x}\right) \approx 0.693\left(2^{x}\right) \\
\hline D_{x}\left(3^{x}\right) \approx 1.0986\left(2^{x}\right) \\
\hline D_{x}\left(2.5^{x}\right) \approx 0.916\left(2^{x}\right) \\
\hline D_{x}\left(\frac{1}{2}^{x}\right) \approx-0.693\left(2^{x}\right) \\
\hline D_{x}\left(\frac{1}{3}^{x}\right) \approx-1.0986\left(2^{x}\right) \\
\hline
\end{array}
$$

Should we memorize these？Impossible！You would never be able to do it，espe－ cially if we added some additional functions such as $4^{x}, 5^{x}$ ，and so on．

Life would be simple if we could locate a base $a$ such that $\lim _{h \rightarrow 0} \frac{a^{h}-1}{h}=1$ ．For such a value of $a$ ，we would have

$$
D_{x}\left(a^{x}\right)=a^{x} \lim _{h \rightarrow 0} \frac{a^{h}-1}{h}=a^{x} \cdot 1=a^{x}
$$

This would be the easiest derivative formula to remember：The derivative of the function would be the function we started with！We would not have to remember anything for this derivative formula．
YOU TRY IT 12．7．OK，try to locate a value of $a$ so that $\lim _{h \rightarrow 0} \frac{a^{h}-1}{h}=1$ ．Estimate this limit using $h=0.000001$ ．From the table above，should you start with $a>3$ ？Between 2.5 and 3？ Less than 2 ？

| Your $a$ |  |  |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\lim _{h \rightarrow 0} \frac{a^{h}-1}{h}$ |  |  |  |  |  |  |


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The exact base that we seek is given a special designation：

DEFINITION 12.1.14. $e$ is the real number such that $\lim _{h \rightarrow 0} \frac{e^{h}-1}{h}=1$. Its approximate value is $e \approx 2.7182818285 \ldots$. Further, $e$ is a transcendental, irrational number (like $\pi$.)

From our earlier work, this means
THEOREM 12.1.15. The function $e^{x}$ is the natural exponential function. It is differentiable (and hence continuous) with

$$
\frac{d}{d x}\left(e^{x}\right)=e^{x}
$$

The number $e \approx 2.7182818285 \ldots$ may seem unnatural at first, but it is actually the most natural choice for a base of an exponential function because its derivative is itself. In other words, at any point along the curve $y=e^{x}$, the slope of the function is the same as its $y$-value.

We can use our new derivative rule in combination with our earlier derivative properties.
EXAMPLE 12.1.16. Determine the derivatives of the following functions.
(a) $f(x)=6 e^{x}+\frac{11}{\sqrt{x^{5}}}-12$
(b) $g(s)=-\frac{e^{s}}{2}-\frac{1}{7 s^{2}}-e$

SOLUTION. In both cases it is convenient to rewrite the functions before taking the derivatives.
(a) Use the sum and constant multiple rules along with power rule and exponential rule.

$$
D_{x}\left(6 e^{x}+\frac{11}{\sqrt{x^{5}}}-12\right)=D_{x}\left(6 e^{x}+11 x^{-5 / 2}-12\right)=6 e^{x}-\frac{55}{2} x^{3 / 2}
$$

(b) Similarly,

$$
D_{s}\left(-\frac{e^{s}}{2}-\frac{1}{7 s^{2}}-e\right)=D_{s}\left(-\frac{1}{2} e^{s}-\frac{1}{7} s^{-2}-e\right)=-\frac{e^{s}}{2}+\frac{2 s^{-3}}{7} .
$$

EXAMPLE 12.1.17. Find the equation of the tangent line to $f(x)=3 e^{x}-e x+2$ at $x=1$.
SOLUTION. We need a point and a slope to determine the equation of the tangent. Since $f(1)=3 e^{1}-e+2=2 e+2$, the point is $(1, f(1))=(1,2 e+2)$. The slope is

$$
m=\left.\frac{d f}{d x}\right|_{x=1}=3 e^{x}-\left.e\right|_{x=1}=2 e
$$

So the equation of the tangent line to $f(x)=3 e^{x}-e x+2$ at $x=1$ is $y-y_{1}=$ $m\left(x-x_{1}\right)$ or

$$
y-(2 e+2)=2 e(x-1) \quad \text { or } \quad y=2 e x+2
$$

## Higher-Order Derivatives

When a function $f(x)$ is differentiable, we obtain a new function $f^{\prime}(x)$. An precisely because $f^{\prime}(x)$ is also a function, we can ask wether $f^{\prime}(x)$ itself is differentiable. That is, does $D_{x}\left(f^{\prime}(x)\right)$ exist? If it does, then the derivative of the derivative of $f$ is called the second derivative of $f(x)$ and is denoted by $f^{\prime \prime}(x)$. In other words,

$$
f^{\prime \prime}(x)=\frac{d}{d x}\left[f^{\prime}(x)\right]=\frac{d}{d x}\left[\frac{d}{d x}(f(x))\right] .
$$

In a similar way, if the derivative of the second derivative of $f(x)$ exists, we call it the third derivative of $f(x)$ and denote it by $f^{\prime \prime \prime}(x)$. We can continue in this manner as long as the derivatives can be calculated.

EXAMPLE 12.1.18. Find the first four derivatives of $f(x)=x^{4}+3 x^{2}+\sqrt{x}+7$.
SOLUTION. Use the general power rule and the general sum and and constant multiple rules to determine the derivatives.

$$
\begin{aligned}
f^{\prime}(x) & =4 x^{3}+6 x+\frac{1}{2} x^{-1 / 2} \\
f^{\prime \prime}(x) & =12 x^{2}+6-\frac{1}{4} x^{-3 / 2} \\
f^{\prime \prime \prime}(x) & =24 x+\frac{3}{8} x^{-5 / 2} \\
f^{\prime(4)}(x) & =24-\frac{15}{16} x^{-7 / 2}
\end{aligned}
$$

Note: To avoid using large numbers of ' symbols, for $n>3$ we use $f^{(n)}(x)$ to denote the $n$-th derivative of $f$, if it exists.

EXAMPLE 12.1.19. Find the first three derivatives of $f(x)=6 e^{x}+2 x^{2}$. What can you say about $f^{(n)}(x)$ for $n>3$ ?

SOLUTION. Use the general power rule, exponential rule, and the general sum and and constant multiple rules to determine the derivatives.

$$
\begin{aligned}
f^{\prime}(x) & =6 e^{x}+4 x \\
f^{\prime \prime}(x) & =6 e^{x}+4 \\
f^{\prime \prime \prime}(x) & =6 e^{x}
\end{aligned}
$$

If $n>3$, then $f^{(n)}(x)=6 e^{x}$.
We have several interpretations for the (first) derivative of a function. The most general interpretation of $f^{\prime}(x)$ is as the rate of change in the quantity $f$ per unit change in $x$. When we interpret this geometrically using the graph of $f$, then $f^{\prime}(x)$ represents the slope of the graph at each $x$. We will see later in the term that $f^{\prime \prime}(x)$ can be interpreted as the 'concavity' or bend of the graph.

Because the derivative of $f$ in general represents the rate of change in the quantity $f$, when $f(x)$ represents the position of an object along a straight line, then $f^{\prime}(x)$ represents the velocity of that same object. Taking the derivative of $f^{\prime}(x)$, we get the rate of change in the velocity of the object. Having driven a car, you know that the rate of change in velocity is acceleration. Thus, for motion problems $f^{\prime \prime}(x)$ represents acceleration. In this same context, the third derivative of the position function is called the jerk of an object. In physics, jerk, also known as jolt, surge, or lurch. (I think the motion sickness you sometimes feel in elevators (or rollercoasters?) is a biological sensation of 'jerk'.)

Other notation
If $y=f(x)$, then $f^{\prime \prime}(x)$ can be indicated in several ways. The most common are

$$
f^{\prime \prime}(x)=f^{(2)}(x)=y^{\prime \prime}=\frac{d^{2} f}{d x^{2}}=\frac{d^{2}}{d x^{2}}(f(x))=\frac{d^{2} y}{d x^{2}}
$$

