The Product and Quotient Rules

Now suppose that f and g are two differentiable functions. We have seen that the derivative rules for the sum, difference, and constant multiples of such functions are straightforward. What about the derivative of the product f(x)g(x)? We might guess that like limits, the derivative of a product is the product of the derivatives. But this turns out not to be the case.

Even a simple example shows that this formula is not true. Let f(x) = g(x) = x. Then $f(x)g(x) = x^2$. So we know how to take the derivatives of all of these functions. In particular,

$$\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}[x^2] = 2x,$$

but

$$f'(x)g'(x) = \frac{d}{dx}[x] \cdot \frac{d}{dx}[x] = 1 \cdot 1 = 1,$$

so comparing, we see

$$D_x[f(x)g(x)] \neq f'(x)g'(x).$$

To determine the correct formula for the derivative of a product, we carefully use the definition of the derivative. So, assume that f and g are differentiable functions. Then by definition,

$$D_x[f(x)g(x)] = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Add 0 to the numerator in the form -f(x)g(x+h) + f(x)g(x+h) to obtain

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

then split the fraction into two limits

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \lim_{h \to 0} \frac{f(x)g(x+h) - f(x)g(x)}{h}$$

then factor each limit

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot g(x+h) + \lim_{h \to 0} f(x) \cdot \frac{g(x+h) - g(x)}{h}$$

and use the product rule for limits

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h \to 0} g(x+h) + \lim_{h \to 0} f(x) \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

and use the definition of the derivative

$$= f'(x) \cdot \lim_{h \to 0} g(x+h) + \lim_{h \to 0} f(x) \cdot g'(x)$$

and finally use that *g* is continuous so we can 'just plug in' h = 0 to evaluate the first limit and there is no *h* in the second—f(x) is constant relative to *h*, so

$$= f'(x)g(x) + f(x) \cdot g'(x).$$

This gives us

THEOREM 12.1.18 (Product Rule for Derivatives). Assume that f and g are differentiable functions. Then the product f(x)g(x) is also differentiable and

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

Let's check that our new derivative formula gives the correct answer to the problem we looked at above. Let f(x) = g(x) = x. Then $f(x)g(x) = x^2$. So if we

use the power rule

$$\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}[x^2] = 2x,$$

and if we use the new product rule

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x) = D_x(x) \cdot x + x \cdot D_x(x) = 1 \cdot x + x \cdot 1 = 2x.$$

We get the same answer whether we same answer whether we use the power rule (much easier) of the product rule. Now let's consider some products where the product rule is the *only* tool that applies that we have developed so far.

EXAMPLE 12.1.19. Determine the derivatives of each of the following functions.

(a) $3x^2e^x$ (b) $\frac{e^x}{6}(x^2-6x+1)$ (c) $(4s^3-1)(2s+5)$ (d) $(3x^2-x)^2$ (e) e^{2t}

SOLUTION. We will need to use the product rule-so we will have to identify the functions *f* and *g*, and then we will need to use athe earlier derivative rules that we developed to find the derivatives of *f* and *g*.

- (a) $D_x(3x^2 e^x) = D_x(3x^2) \cdot e^x + 3x^2 \cdot D_x(e^x) = \underbrace{\frac{f'g}{6xe^x} + \frac{fg'}{3x^2e^x}}_{fg'} = (3x^2 + 6x)e^x.$
- (b) This time

f

$$D_x \left[\underbrace{\frac{e^x}{6}}_{f} \underbrace{(x^2 - 6x + 1)}_{g} \right] = D_x \left(\frac{e^x}{6} \right) \cdot (x^2 - 6x + 1) + \frac{e^x}{6} \cdot D_x (x^2 - 6x + 1)$$
$$= \underbrace{\frac{f'g}{6} (x^2 - 6x + 1)}_{f} + \underbrace{\frac{e^x}{6} (2x - 6)}_{f}$$
$$= \frac{e^x}{6} (x^2 + 2x - 5).$$

(c) For the derivative of $(4s^3 - 1)(2s + 5)$, we could actually multiply the two functions first and then take the derivative of the resulting polynomial. (Try it and compare to the answer below.) However, we will continue to use the product rule.

$$D_{s}\left[\overbrace{(4s^{3}-1)}^{f}\overbrace{(2s+5)}^{g}\right] = D_{s}\left(4s^{3}-1\right) \cdot (2s+5) + (4s^{3}-1)D_{s}(2s+5)$$
$$=\overbrace{12s^{2}(2s+5)}^{f'g} + \overbrace{(4s^{3}-1)(2)}^{fg'}$$
$$= 32s^{3} + 60s^{2} - 2.$$

(*d*) For the derivative of $(3x^2 - x)^2$, we can convert the squaring function to a product of two polynomials and use the product rule for derivatives or we could performing the squaring operation first and then take the derivative. (Try the latter method; what do you get?). Notice that instead of f and g we now have *f* and *f*.

$$D_x \Big[(3x^2 - x)^2 \Big] = D_x \Big[\underbrace{(3x^2 - x)}_{f} \underbrace{(3x^2 - x)}_{f} \Big] = D_x \Big(3x^2 - x \Big) \cdot (3x^2 - x) + (3x^2 - x)D_x (3x^2 - x)) \\ = \underbrace{\frac{f'f}{(6x - 1)(3x^2 - x)}}_{2f'f} + \underbrace{\frac{ff'}{(3x^2 - x)(6x - 1)}}_{2f'f} \\ = \underbrace{\frac{2f'f}{36x^3 - 18x^2 + 2x}}_{2f'f}.$$

By looking above the braces, this example shows that $D_x([f(x)]^2) = 2f'(x)f(x)$. Shortly we will prove a general power rule for derivatives of the form $D_x([f(x)]^n)$.

(e) For the derivative of e^{2t} , we can write the function as a product $[f(t)]^2 = e^t e^t$. (Shortly we will see how to calculate $D_t(e^{2t})$ directly.) For the moment, then we can use the fact we demonstrated in pat (d) that $D_t([f(t)]^2) = 2f'(t)f(t)$.

$$D_t\left\lceil \overbrace{(e^{2t})}^{f^2} \right\rceil = \overbrace{2D_t(e^t)}^{2f'} \cdot \overbrace{(e^t)}^{f} = \overbrace{2(e^t)(e^t)}^{2f'f} = 2e^{2t}.$$

Let's generalize this last example. We now know that if $g(x) = e^{2x}$, then $D_x[g(x)] = D_x[e^{2x}] = 2e^{2x}$. We can use this to determine $D_x[e^{3x}]$ by rewriting e^{3x} as the product $e^x e^{2x}$. Then

$$D_x[e^{3x}] = D_x[\underbrace{f_{e^x}}_{e^x}, \underbrace{f_{e^{2x}}}_{e^{2x}}] = D_x(e^x) \cdot e^{2x} + e^x D_x(e^{2x}) = \underbrace{e^x \cdot e^{2x}}_{e^x} + \underbrace{e^x \cdot 2e^{2x}}_{e^x} = 3e^{3x}.$$

Now that we know the derivative of e^{3x} , we can find the derivative of e^{4x} by using the product $e^x \cdot e^{3x}$.

$$D_x[e^{4x}] = D_x[\overbrace{e^x}^{f} \cdot \overbrace{(e^{3x})}^{g}] = D_x(e^x) \cdot e^{3x} + e^x D_x(e^{3x}) = \overbrace{e^x \cdot e^{3x}}^{f'g} + \overbrace{e^x \cdot 3e^{3x}}^{fg'} = 4e^{4x}.$$

Take a look at the pattern: $D_x[e^{2x}] = 2e^{2x}$, $D_x[e^{3x}] = 3e^{3x}$, and $D_x[e^{4x}] = 4e^{4x}$. Using the same method as above, we can show that

COROLLARY 12.1.20. If *k* is a positive integer, then $D_x[e^{kx}] = ke^{kx}$. In fact, a more general result is true: If *r* is any *real number*, then $D_x[e^{rx}] = re^{rx}$.

EXAMPLE 12.1.21. Determine the derivative of $f(x) = 6e^{-3x/2}$.

SOLUTION. Using Corollary 12.1.20, we find $D_x(e^{-3x/2}) = -\frac{3}{2} \cdot 6e^{-3x/2} = -9e^{-3x/2}$.

Derivatives of Quotients

We have developed a rule for taking derivatives of products of differentiable functions so it is natural to ask whether there is a rule to help us take the derivative of the quotient of two functions $\frac{f(x)}{g(x)}$. The simplest such quotient is a reciprocal of a differentiable function, $\frac{1}{g(x)}$. Let's see if we can determine $\frac{d}{dx} \left[\frac{1}{g(x)} \right]$, assuming that g(x) is differentiable and that $g(x) \neq 0$. Using the definition of the derivative, we have

$$\frac{d}{dx} \left[\frac{1}{g(x)} \right] = \lim_{h \to 0} \frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} = \lim_{h \to 0} \frac{\frac{g(x) - g(x+h)}{g(x+h)g(x)}}{h}$$
$$= \lim_{h \to 0} \frac{g(x) - g(x+h)}{h} \cdot \frac{1}{g(x+h)g(x)}$$

now factor out -1

$$=\lim_{h\to 0} -\frac{g(x+h)-g(x)}{h}\cdot\frac{1}{g(x+h)g(x)}$$

use the definition of the derivative to evaluate the first part of the limit and use continuity to evaluate the second part

$$= -g'(x) \cdot \frac{1}{(g(x))^2}.$$

We have proven the following result.

THEOREM 12.1.22. If g(x) is differentiable, then so is $\frac{1}{g(x)}$ where $g(x) \neq 0$. Further,

$$\frac{d}{dx}\left[\frac{1}{g(x)}\right] = -\frac{g'(x)}{(g(x))^2}.$$

This is a rule that you should *not* memorize. We will prove a more useful general rule shortly.

EXAMPLE 12.1.23. We have proven that $\frac{d}{dx}[x^n] = nx^{n-1}$, where *n* is a *positive* integer. We then stated, but did not prove that the same power rule applied to negative powers. Now that we have a reciprocal rule for derivatives, this is easy to demonstrate. Using Theorem 12.1.22

$$\frac{d}{dx}\left[x^{-n}\right] = \frac{d}{dx}\left[\frac{1}{x^n}\right]^{\text{Theorem 12.1.22}} - \frac{nx^{n-1}}{(x^n)^2} = -\frac{nx^{n-1}}{x^{2n}} = -\frac{n}{x^{n+1}} = -nx^{-n-1}.$$

This is exactly what we claimed in the general power rule, Theorem 12.1.8.

EXAMPLE 12.1.24. Determine the derivative of $\frac{1}{e^{2x}+3x^2}$.

Solution. Using Theorem 12.1.22, we think of $g(x) = e^{2x} + 3x^2$ and $g'(x) = 2e^{2x} + 6x$, so

$$D_x\left[\frac{1}{e^{2x}+3x^2}\right] = -\frac{g'(x)}{(g(x))^2} = -\frac{2e^{2x}+6x}{(e^{2x}+3x^2)^2}.$$

YOU TRY IT 12.8. Let *k* be a positive integer. Determine the derivative of $g(x) = e^{-kx}$ using Theorem 12.1.22 and Corollary 12.1.20.

Answer to you try it 12.8. $D_x(e^{-\kappa x}) = -ke^{-\kappa x}$.

Let's turn now the general problem of derivatives of quotient functions. Suppose that *f* and *g* are differentiable and that $g(x) \neq 0$. To determine $D_x\left(\frac{f(x)}{g(x)}\right)$ we can think of the quotient as a product using a reciprocal. Using both the product and reciprocal rules,

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x)\cdot\frac{1}{g(x)}\right) = f'(x)\cdot\frac{1}{g(x)} + f(x)\cdot\left(-\frac{g'(x)}{(g(x))^2}\right) \\ = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{(g(x))^2} \\ = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

In other words, we have shown

THEOREM 12.1.25 (Quotient Rule for Derivatives). Assume that f and g are differentiable and that $g(x) \neq 0$. Then $\frac{f(x)}{g(x)}$ is differentiable and

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

Note: Though we computed the general formula for the quotient rule using the reciprocal rule for derivatives, in general you should not do a the derivative of a *particular* quotient that way. Instead, make use of the quotient rule directly.

EXAMPLE 12.1.26. Determine the derivatives of the following functions, where they are defined.

(a)
$$\frac{x^3-2}{x^3+2}$$
 (b) $\frac{2x+4}{3x-5}$ (c) $\frac{x^2+1}{e^{2x}}$ (d) $\frac{x^4+2x}{x^2+1}$

Solution. Use the quotient rule for each.

(a) Here $f(x) = x^3 - 2$ and $g(x) = x^3 + 2$. $D_x \left[\frac{x^3 - 2}{x^3 + 2} \right] = \frac{3x^2(x^3 + 2) - (x^3 - 2)3x^2}{(x^3)^2} = \frac{3(x^3 + 2) - (x^3 - 2)3}{x^4}$ $= \frac{3x^3 + 6 - (3x^3 - 6)}{x^4}$ $= \frac{12}{x^4}.$

(*b*) This time f(x) = 2x + 4 and g(x) = 3x - 5.

$$D_x \left[\frac{2x+4}{3x-5} \right] = \frac{2(3x-5) - (2x+4)3}{(3x-5)^2} = \frac{6x-10 - (6x+12)}{(3x-5)^2} = -\frac{22}{(3x-5)^2}$$

(c) Here $f(x) = x^2 + 1$ and $g(x) = e^{2x}$.

$$D_x\left[\frac{x^2+1}{e^{2x}}\right] = \frac{2xe^{2x}) - (x^2+1)2e^{2x}}{(e^{2x})^2} = \frac{2x-2x^2-2}{e^{4x}} = \frac{-2x^2+2x-2}{e^{4x}}.$$

(d) Lastly, $f(x) = x^4 + 2x$ and $g(x) = x^2 + 1$.

$$D_x \left[\frac{x^4 + 2x}{x^2 + 1} \right] = \frac{(4x^3 + 2)(x^2 + 1) - (x^4 + 2x)2x}{(x^2 + 1)^2}$$
$$= \frac{(4x^5 + 4x^3 + 2x^2 + 2) - (2x^5 + 4x^2)}{(x^2 + 1)^2}$$
$$= \frac{2x^5 + 4x^3 - 2x^2 + 2}{(x^2 + 1)^2}.$$

EXAMPLE 12.1.27. The product and quotient rules may both be required in a single problem. Determine the derivative of $y = \frac{x+1}{x^2 e^x}$.

Solution. Here f(x) = x + 1 and $g(x) = x^2 e^x$. When taking the derivative of g, we must use the product rule.

$$D_x \left[\frac{x+1}{x^2 e^x} \right] = \frac{1(x^2 e^x) - (x+1)[2xe^x + x^2 e^x]}{(x^2 e^x)^2}$$
$$= \frac{x^2 e^x - [2x^2 e^x + x^3 e^x + 2xe^x + x^2 e^x]}{(x^2 e^x)^2}$$
$$= \frac{x^2 e^x - 2x^2 e^x - x^3 e^x - 2xe^x - x^2 e^x}{(x^2 e^x)^2}$$
$$= \frac{-2x^2 e^x - x^3 e^x - 2xe^x}{(x^2 e^x)^2}$$
$$= \frac{-(xe^x)(2x + x^2 + 2)}{x^4 e^{2x}}$$
$$= \frac{-(x^2 + 2x + 2)}{x^3 e^x}.$$

Higher-Order Derivatives

When a function f(x) is differentiable, we obtain a new function f'(x). And precisely because f'(x) is also a function, we can ask wether f'(x) *itself* is differentiable. That is, does $D_x(f'(x))$ exist? If it does, then the derivative of the derivative of f is called the **second derivative** of f(x) and is denoted by f''(x). In other words,

$$f''(x) = \frac{d}{dx}[f'(x)] = \frac{d}{dx}\left[\frac{d}{dx}(f(x))\right].$$

EXAMPLE 12.1.28. Find the second derivatives of $f(x) = x^4 + 3x^2 - \sqrt{x} + 7$ and $g(x) = 23x^2 - 3x^{-2} + 11x$.

Solution. To determine the second derivative of a function we must first determine the first derivative. Using the sum and power rules we see that

$$f'(x) = D_x[x^4 + 3x^2 - \sqrt{x} + 7] = 4x^3 + 6x - \frac{1}{2}x^{-1/2}$$

so

$$f''(x) = D_x \left[4x^3 + 6x - \frac{1}{2}x^{-1/2} \right] = 12x^2 + 6 + \frac{1}{4}x^{-3/2}$$

Similarly

$$g'(x) = D_x[23x^2 - 3x^{-2} + 11x] = 26x + 6x^{-3} + 11x$$

so

$$g''(x) = D_x \left[26x + 6x^{-3} + 11 \right] = 26 - 18x^{-4}$$

There is no reason to stop with the second derivative. Each time we compute a derivative, we can ask whether the new function is also differentiable and compute the third, fourth, or other *higher-order* derivatives, if possible. The third derivative would be denoted by f'''(x). Since it is count the number of primes in a higher derivative, we use $f^{(4)}(x)$ to indicate the fourth derivative of f(x). More generally, we denote the *n*th derivative of *f* by $f^{(n)}$, where *n* is a positive integer.

EXAMPLE 12.1.29. Find the first three derivatives of $f(x) = 6e^x + 2x^2$. What can you say about $f^{(n)}(x)$ for n > 3?

SOLUTION. Using the derivative rules we have developed,

$$f'(x) = 6e^{x} + 4x$$
$$f''(x) = 6e^{x} + 4$$
$$f'''(x) = 6e^{x}$$

If n > 3, then $f^{(n)}(x) = 6e^x$.

YOU TRY IT 12.9. Find the first four derivatives of $f(x) = x^4 + 3x^2 + \sqrt{x} + 7$.

$$\frac{3}{3}x^{-5/2}; f^{I(4)}(x) = 24 - \frac{15}{16}x^{-7/2}.$$
Answer to you try it 12.9. $f^{I}(x) = 4x^3 + 6x + \frac{1}{2}x^{-1/2}; f^{II}(x) = 12x^2 + 6 - \frac{1}{4}x^{-3/2}; f^{III}(x) = 24x + \frac{1}{2}x^{-7/2}.$

YOU TRY IT 12.10. Let $f(x) = 2e^x + 4x^2$. Determine the first four derivatives of f and then determine a general formula for $f^{(n)}(x)$ for $n \ge 4$.

Answer to you try it 12.10. $f'(x) = 2e^x + 8x$; $f''(x) = 2e^x + 8$; $f''(x) = 2e^x$. Since $f'''(x) = 2e^x$. In other words, $f'''(x) = f^{(4)}(x) = 2e^x$ for any integer $n \ge 3$.

YOU TRY IT 12.11. Let $f(x) = e^{kx}$, where k is a constant. Determine a general formula for $f^{(n)}(x)$.

Answer to you try it is. Using Corollary ??, we find that $f'(x) = k_e^{kx}$, $f''(x) = k^2e^{kx}$, $f'''(x) = k^3e^{kx}$, $f'''(x) = k^3e^{kx}$, and more generally, $f'(n)(x) = k^ne^{kx}$.

YOU TRY IT 12.12. Let $f(x) = x^n$, where *n* is a positive integer. With the same *n* determine the formula for $f^{(n)}(x)$.

The number $n(n_1)(n-2)\cdots(2)(1)$ is called **n factorial** and is denoted by n!.

$$f^{(n)}(x) = \frac{1}{n(n_1)(n-2)\cdots(2-n)(n-1)} = n^{-n}x \underbrace{(1)(2)\cdots(2-n)(n-1)(n-1)}_{n(n-1)(n-1)(n-1)} = (1)(1)$$

ANSWER TO YOU TRY IT 12.11. Using the power rule, we find that $f'(x) = nx^{n-1}$, $f''(x) = n(n-2)x^{n-3}$. Continuing a total of n times, $1)x^{n-2}$, $f'''(x) = n(n-1)(n-2)x^{n-3}$.

Other notation. As with first derivatives, be aware that other notation exists for higher-order derivatives. For example, if y = f(x) is a function whose second derivative exists, then any of the following may be used to indicate this:

$$f''(x) = f^{(2)}(x) = \frac{d^2f}{dx^2} = \frac{d^ny}{dx^n}.$$

Similarly, for the n^{th} derivative of f, we may write

$$f^{(n)}(x) = \frac{d^n f}{dx^n} = \frac{d^n y}{dx^n}.$$

Why bother? Why do we bother with higher derivatives? Depending on the context, higher order derivatives have a variety of interpretations. Remember that any derivative, properly interpreted, represents a rate of change. For example if s(t) represents the position of an object moving along a straight line path at time t, we have seen that s'(t) = v(t) represents the velocity of this same object. Consequently, s''(t) = v'(t) represents the rate of change in the velocity, that is s''(t) is the *acceleration* of the object. You use this language informally when driving a car: You use the *accelerator* to change the velocity of your car.

The derivative of acceleration is called *jerk* and is sometimes denoted by j(t). Thus, jerk

$$j(t) = a'(t) = v''(t) = s'''(t).4$$

Excessive jerk may result in an uncomfortable ride on elevators and engineers expend considerable design effort to minimize "jerky motion." As an everyday example, driving in a car can show effects of acceleration and jerk. The more experienced drivers accelerate smoothly, but beginners provide a jerky ride. Riding with an inexperienced driver changing gears with a foot-operated clutch lets you experience severe jerk.

Listening to the news, you may hear statements such as, "The rate of growth of the economy is slowing." The rate of growth of the economy is determined by a first derivative of an appropriate function. When we say that the rate of growth is slowing, this represents a second derivative. The economy is still growing (first derivative), but at a slower rate than it was (second derivative).

Later in the semester we will explore the geometric meaning of the first and second derivatives of a function. We will see that the first derivative tells us whether the graph of the function is rising or falling while the second derivative tells us whether the curves is bending up or down. Using this information and a few critical points, we can obtain a rough sketch of a curve that exhibits all of its key behavior.

YOU TRY IT 12.13. Each of the functions graphed below is increasing. In which of the functions is the rate of growth increasing?



ANSWER TO YOU TRY IT 12.13. In graph (b). As one moves from left to right, the graph rises more quickly.