Key Idea: One-to-one Functions

More complete notes are online at the course website.

DEFINITION 21.1.1. A function f is **one-to-one** if whenever $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. This means that f never has the same output value twice.

☞ You should memorize this definition.

EXAMPLE 21.1.2. The function on the left is NOT one-to-one, because $f(x_1) = f(x_2)$ even though $x_1 \neq x_2$. Similarly $f(x_3) = f(x_4)$ even though $x_3 \neq x_4$.

The function on the right is one-to-one; it never has the same output value twice.

EXAMPLE 21.1.3. Show that the function $f(x) = \sqrt{x^2 + 9}$ is not one-to-one.

> **SOLUTION.** Using the definition of one-to-one, we need to find two different values $x_1 \neq x_2$ so that $f(x_1) = f(x_2)$. We know that squaring produces the same output for inputs a and $-a$. So if we use $x_1 =$ ____________ and $x_2 =$ ____________, then
>
> $f(x_1) =$ ________________________________ and $f(x_2) =$ ________________________________.
>
> We get the same output twice, so f is not one-to-one.

Look back at the graphs in Example 21.1.2. The reason that the first function is not one-to-one is that a horizontal line meets the graph twice. The horizontal line represents a particular y-value. If it meets the graph twice, there must be two different x-values that have the same y-value. So we have the following theorem.

YOU TRY IT 21.1. Show that the function $f(x) = x^2 + \cos x$ is not one-to-one.

THEOREM 21.1.4 (Horizontal Line Test, HLT). A function is one-to-one if and only if no horizontal line meets the graph more than once.

YOU TRY IT 21.2. Draw a function that passes the HLT and one that fails it.

Inverse Functions

DEFINITION 21.1.5. A function g is the **inverse** of the function f if

1. $g(f(x)) = x$ for all x in the domain of f
2. $f(g(x)) = x$ for all x in the domain of g

In this situation g is denoted by f^{-1} and is called "f inverse."

In other words, g undoes f and f undoes g. The key fact is that

THEOREM 21.1.6. f has an inverse $\iff$ f is one-to-one $\iff$ f passes the HLT.

The Graph of f^{-1}

Now suppose that $y = f(x)$ has an inverse $f^{-1}(x)$, and assume that a is in the domain of f and that $f(a) = b$. Then using the definition of inverse:

$$f^{-1}(b) \overset{f(a)=b}{=} f^{-1}(f(a)) \overset{\text{Inverse}}{=} a$$

In other words

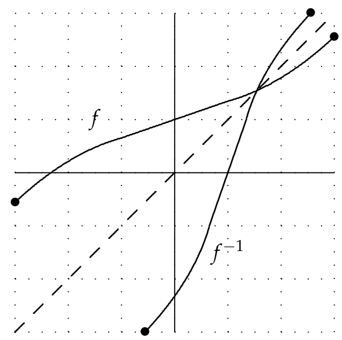
$$\boxed{f(a) = b \iff f^{-1}(b) = a}$$

or

$$\boxed{(a, b) \text{ on the graph of } f \iff (b, a) \text{ is on the graph of } f^{-1}}$$

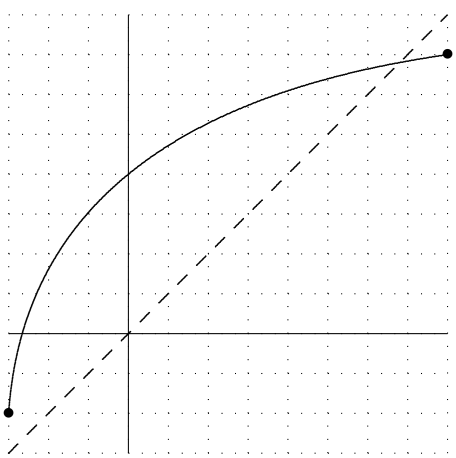
In other words, f and f^{-1} have their x and y coordinates switched. And because the x and y coordinates are switched.

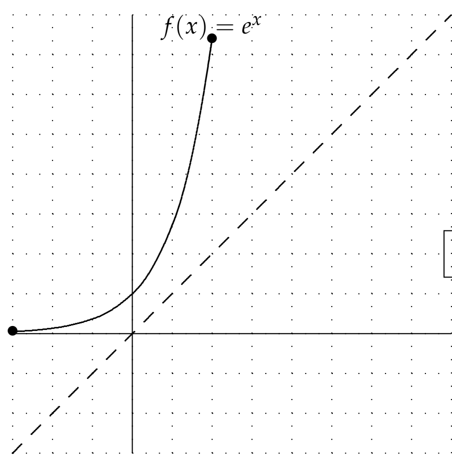
- Domain of f^{-1} = Range of f
- Range of f^{-1} = Domain of f



Thus, if f is one-to-one, we can obtain the graph of f^{-1} by interchanging the x and y coordinates. If we draw the diagonal line $y = x$ and use it as a mirror, notice that the x and y axes are reflected into each other across the line. This is just another way of saying that the x and y coordinates have been switched. So to obtain the graph of f^{-1} all we need to do is to reflect the graph of f in the diagonal line $y = x$, as shown in the margin.

YOU TRY IT 21.3. Draw the graph of f^{-1} for ecah of the functions f graphed below.





$\boxed{f^{-1}(x) = \ln x}$

The Inverse of the Exponential Function e^x

Now consider $y = f(x) = e^x$. We have seen that this function is increasing and that it appears to pass the HLT, so it has an inverse. Using the graph of e^x, we can draw the graph of its inverse. The inverse of the exponential function is the **natural log** function and is denoted by $y = \ln x$. Using the definition of inverse:

$$f(a) = b \iff f^{-1}(b) = a$$
$$e^a = b \iff \ln b = a$$

This means that logs are exponents of e (note where a is in the last line).
☞ What about the derivative of $\ln x$?

Because logs are exponents, logs have the following very useful properties:

(1) $\ln(xy) = \ln x + \ln y$

(2) $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

(3) $\ln(x^r) = r \ln x$