Introduction

Why Calculus?

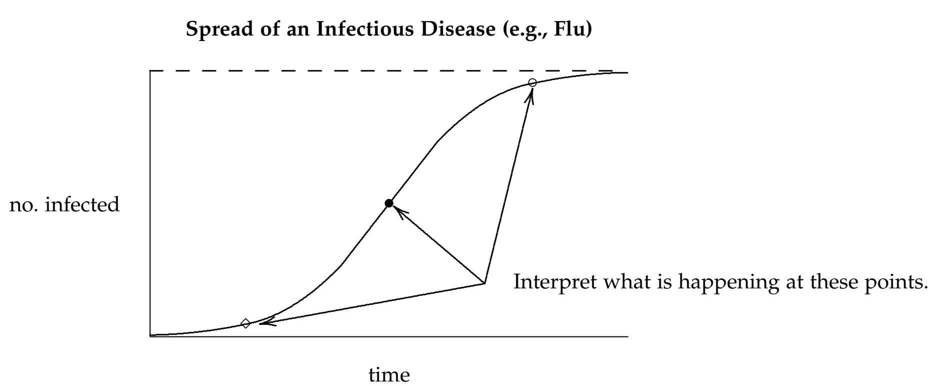
Why are you studying calculus? The answer I hear most often is, "Because I have to! It is required for my major." Several disciplines do require or strongly suggest taking calculus, among them biology, physics, chemistry, architecture, economics, geology, environmental studies, and medicine. Why do these disciplines encourage you to take calculus?

Understanding the Behavior of Functions

The main reason these disciplines require calculus is to understand the behavior of functions. Functions can be used to provide a compact, symbolic description of some aspect of the real world—they act as **models** of various real-world phenomena that these disciplines study and try to understand.

EXAMPLE 0.0.1. Here's a type of function (called a logistic function) that might arise in medical research.

Figure 1: Interpret what is happening to the spread of the disease at each of the marked points. Is it the same everywhere?



In terms of the disease, how would you interpret what is happening near $\bullet$? Near $\diamond$? Near $\circ$? When might medical intervention be most effective? The same graph might also represent

- the 'spread' of information or rumor;
- population growth where resources are limited;
- sales of a new product.

Notice that the curve is characterized by three different rates of change... this is reflected in the steepness or **slope of the curve**. Understanding this notion of 'steepness' or slope is what first term calculus is all about.

We can look at the curve in a purely geometric fashion, analyze it mathematically, then translate this analysis back to the real world situation.

Two Basic Questions and Their Translations

The ability to translate or 'interpret' a process from the real world to mathematics where the analysis is done and then translate the 'solution' back to the real world is what makes mathematics so powerful and important to other disciplines. Here are two key questions that are often 'translated' to other disciplines.

The Slope Problem

We know what it means to talk about the slope of a line, namely

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}}.$$

But what does the 'slope of a curve' mean? How do we determine the value once we make sense of the idea?

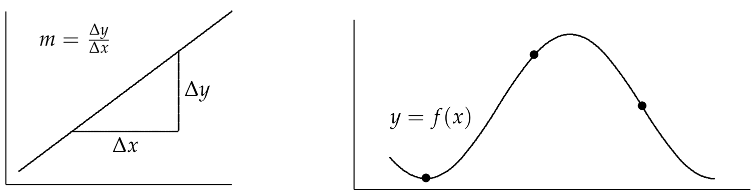


Figure 2: Left: The slope of a line is the same at any point along the line. Right: The slope of a curve varies at each of the points. Estimate the slope at each of the three marked points.

Notice that the slope changes along a curve, unlike along a line. Sometimes the slope is positive: The curve is 'uphill' as one moves from left to right. Other times it is negative where the curve is 'downhill.' What does the curve look like when the slope is 0? It turns out that if you know the slope everywhere you can determine the shape of the curve.

Since all we know how to calculate are slopes of lines, we *must* use the slopes of lines to determine the slope of a curve. We can think of it this way: At each point along the curve we are trying to find the line that has the same slope as the curve does at the point in question. This is called the **tangent line** to the curve at the specified point. Try drawing a few such lines on the curve above. When is the tangent line horizontal?

More generally, to find the slope of a curve at $(x_1, f(x_1))$ we could take a nearby point $(x_2, f(x_2))$ on the curve and compute the slope of the secant[1] line that passes through the two points.

[1] A **secant line** to a curve is a line that passes through two distinct points on the curve.

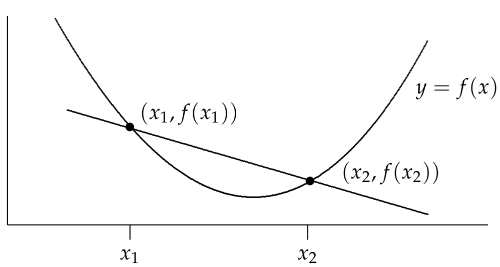


Figure 3: A secant line that passes through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. The slope of this line (roughly) approximates the slope of the curve at the point $(x_1, f(x_1))$.

This secant slope (see Figure 3) provides an *approximation* to the slope of the curve at the point $(x_1, f(x_1))$.

$$\text{Slope at } x_1 \approx \frac{\Delta y}{\Delta x} \approx \frac{y_2 - y_1}{x_2 - x_1}. \tag{1}$$

This approximation obviously depends on the second point that is chosen. So how do we decide which point x_2 to choose?

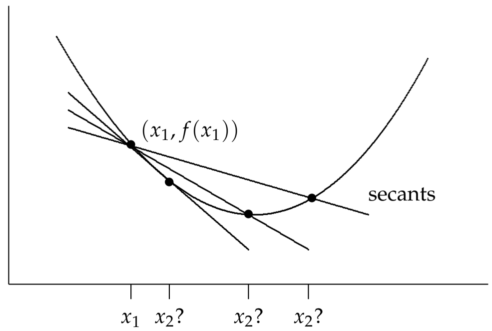


Figure 4: As x_2 approaches x_1, the secant slopes better approximate the slope of the curve right at x_1.

As Figure 4 indicates, the secant approximation to the slope of the curve seems to improve if we take x_2 closer to x_1. Ok, let's let x_2 approach x_1. As x_2 approaches x_1 we see that $y_2 = f(x_2)$ approaches $y_1 = f(x_1)$ (see Figure 4 again). To calculate the *exact slope* of the curve right at x_1 all we need to do is see what happens when x_2 reaches x_1. But in our moment of triumph we find that letting $x_2 = x_1$ in (1) we get

$$\text{Slope at } x_1 = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_1}{x_1 - x_1} = \frac{0}{0} = ?? \tag{2}$$

This is a meaningless expression since we can't divide by 0. This *is* a big problem!

Interpretation: Slope Problem = Instantaneous Rate Problem

Let's think about rates of change. Again, we know how to compute an *average* rate. For example, suppose we drive 130 miles in 2 hours. Then our average velocity is

$$\text{Ave rate} = \text{Ave velocity} = \frac{\Delta\text{distance}}{\Delta\text{time}} = \frac{130}{2} = 65\,\text{mph}. \tag{3}$$

We see that average velocity is the same calculation as the slope of a line, where $\Delta\text{distance} = \Delta y$ and $\Delta\text{time} = \Delta t$. This is another example of the interpretation process we just mentioned.

Suppose that during the trip you glance at the speedo and it says 69 mph. What does that mean? What does an *instantaneous rate* mean? There's a problem: In an instant there is no passage of time ($\Delta t = 0$) and there is no change in position ($\Delta\text{distance} = 0$) so

$$\text{Inst rate} = \text{Inst velocity} = \frac{\Delta\text{distance}}{\Delta\text{time}} = \frac{0}{0} = ??. \tag{4}$$

This is the same undefined mathematical expression that we faced in (2). This a real problem that we will need to solve if we are ever going to make sense out of the notion of an instantaneous rate of change.

Look back and compare (1) and (3) and then (2) and (4). We have the following equivalences:

$$\boxed{\text{Slope of a line} = \text{Average rate of change} = \frac{\Delta y}{\Delta x}} \tag{5}$$

while

$$\boxed{\text{Slope of a curve} = \text{Instantaneous rate of change.}} \tag{6}$$

Think about it: Suppose that you somehow solved the problem of calculating instantaneous rates of change. Do you see how you might use such rates of change to find the maximum value of a function. Hint: How is the function changing just before it reaches its maximum? Just after? So what might be happening to the change right at a maximum?

The Accumulation Problem

There's a second fundamental problem of introductory calculus that we will begin to discuss at the end of this term. It is often described as the 'accumulation problem.' A simple example will illustrate the idea.

Suppose that the Colleges' monitor the water usage on campus and find that during a typical day the instantaneous rate water use in gallons per hour is given by the graph in Figure 5.

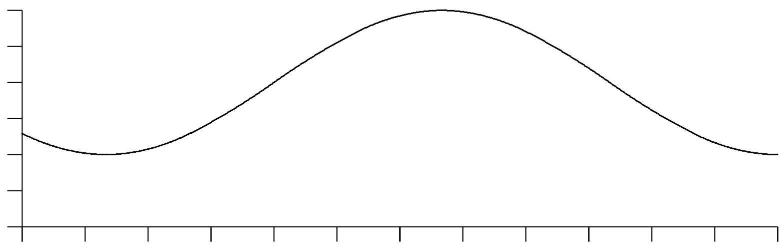


Figure 5: The *rate* at which water is used throughout during a 24-hr period. How *much* water is used during the period?

The question now is given the *rate* of water usage, can we determine the total *amount* of water that was used during the day. Notice that this is almost the reverse of the question we asked earlier. Here we know the instantaneous (flow) rate; what we want to know is the accumulated usage.

If the rate were constant, then we know from previous courses

$$\boxed{\text{rate} \times \text{time} = \text{`accumulation'},} \tag{7}$$

or in our case the amount of water used. If the constant rate were the velocity of a car, then rate $\times$ time would be the distance travelled. But how do we deal with rates that are changing? This is primarily a Calculus II question, but the answer is connected to our original question by the so-called Fundamental Theorem of Calculus (see below). The question has a geometric solution: The accumulation is just the area under the rate curve in Figure 5.

THEOREM 0.0.2 (The Fundamental Theorem of Calculus). Assume that f is continuous on $[a, b]$ and that F is an antiderivative of f on $[a, b]$. Then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

The amazing thing is that by the end of the term you will know what all the symbolism and terminology means!