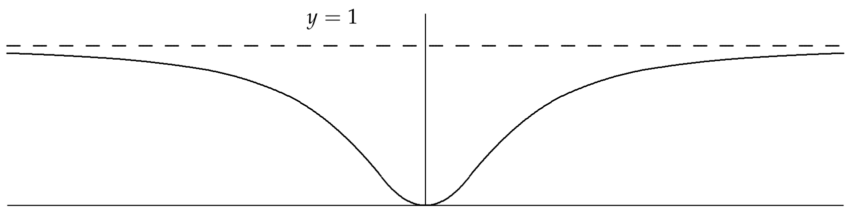
Limits at Infinity

Horizontal Asymptotes

Consider the function $f(x) = \frac{x^2}{x^2+1}$ which is graphed below. It does not have any vertical asymptotes but it does have a horizontal asymptote at $y = 1$.



So what do we mean when we say that f has a horizontal asymptote at $y = 1$? Something like

$$\text{“As } x \text{ gets big, or as } x \to \infty, f(x) \text{ gets close to } y = 1\text{"}$$
$$\text{“As } x \to -\infty, f(x) \text{ gets close to } y = 1\text{"}$$

We can see this by looking at a table of values:

x	± 10	± 100	$\pm 1,000$
$\frac{x^2}{x^2+1}$	1.001	1.0001	1.000001

The following informal definition will be sufficient for our purposes.

DEFINITION 7.0.1 (Limits at Infinity). We say that

$$\lim_{x \to +\infty} f(x) = L$$

if we can make $f(x)$ arbitrarily close to L by taking x sufficiently large. Similarly,

$$\lim_{x \to -\infty} f(x) = M$$

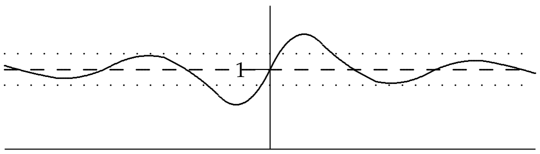
if we can make $f(x)$ arbitrarily close to M by taking x sufficiently large in magnitude but negative.

DEFINITION 7.0.2. The line $y = L$ is a **horizontal asymptote** (HA) for the graph of $f(x)$ if either $\lim_{x \to +\infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$.

Pictures

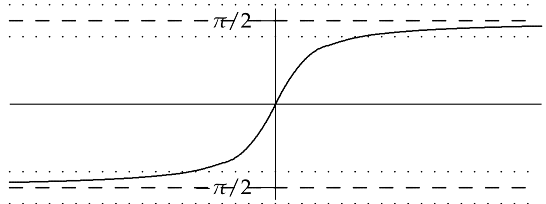
Here are a few graphs of functions with horizontal asymptotes. Notice that the function can cross a horizontal asymptote (but not a vertical one). Notice that if

$y = L$ is a horizontal asymptote, then it means when x is large enough (in one direction or the either) the function stays within a little horizontal corridor about the line $y = L$... i.e., $f(x)$ gets close to L.



$\lim_{x\to\infty} f(x) = 1$ so $y = 1$ is an HA for $y = f(x)$

$\lim_{x\to-\infty} = \arctan x = -\pi/2$ and $\lim_{x\to\infty} = \arctan x = \pi/2$



$y = \pm\pi/2$ are HA's for $y = \arctan x$

Working with Limits at Infinity

Here's a simple function $f(x) = \frac{1}{x}$. As x gets large in magnitude it is easy to see that $f(x)$ approaches 0.

x	± 10	± 100	$\pm 1,000$	$\pm 1,000,000$
$\frac{1}{x}$	± 0.1	± 0.01	± 0.001	± 0.000001

In other words we have the following

FACT 7.1. Let $f(x) = \frac{1}{x}$, then

$$\lim_{x\to\infty} \frac{1}{x} = 0 \text{ and } \lim_{x\to-\infty} \frac{1}{x} = 0.$$

This means that $y = 0$ is an HA for $f(x) = \frac{1}{x}$.

Most of the basic limit laws carry over for limits at infinity. So we can use them to show: if $r > 0$ (even fractional values of r are fine), then

$$\lim_{x\to\infty} \frac{c}{x^r} \stackrel{\text{Const Mult}}{=} c \cdot \lim_{x\to\infty} \frac{1}{x^r} \stackrel{\text{Power}}{=} c\left(\lim_{x\to\infty} \frac{1}{x}\right)^r = c(0)^r = 0.$$

FACT 7.2. If $r > 0$, then

$$\lim_{x\to\infty} \frac{c}{x^r} = 0.$$

Likewise, as long as x^r is defined when $x < 0$, then

$$\lim_{x\to-\infty} \frac{c}{x^r} = 0.$$

This makes sense, just think about it: If x gets large, then x^r gets large (if $r > 0$) so $\frac{c}{x^r}$ gets small.

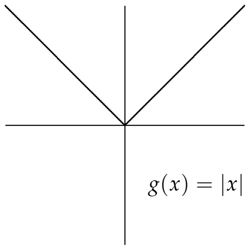
EXAMPLE 7.0.3. Determine $\lim_{x\to\infty} \frac{7x^2+1}{6x^2-3x}$.

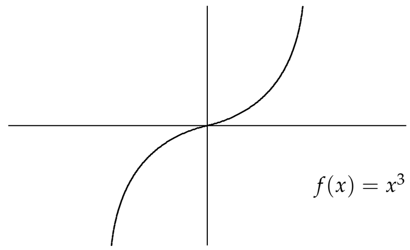
Solution. If we divide both the numerator and denominator by the highest power of x in the denominator, things quickly simplify.

$$\lim_{x\to\infty} \frac{7x^2+2}{6x^2-3x} \stackrel{\div x^2}{=} \lim_{x\to\infty} \frac{7+\frac{2}{x^2}}{6-\frac{3}{x}} \stackrel{\text{Fact 7.2}}{=} \frac{6+0}{7-0} = \frac{6}{7}.$$

7.1 *Infinite Limits at Infinity*

Many simple and familiar functions get very large in magnitude as x itself gets large in magnitude We say that such functions have an **infinite limit at infinity**. A couple of familiar examples include $g(x) = |x|$ and $f(x) = x^3$ illustrated below.





DEFINITION 7.1.1 (Infinite Limits at Infinity). If $f(x)$ becomes arbitrarily large as x becomes arbitrarily large, then we write

$$\lim_{x\to\infty} f(x) = \infty.$$

If $f(x)$ becomes arbitrarily large in magnitude but negative as x becomes arbitrarily large, then we write

$$\lim_{x\to\infty} f(x) = -\infty.$$

Similar definitions are used for $\lim_{x\to-\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

The End Behavior of Polynomials

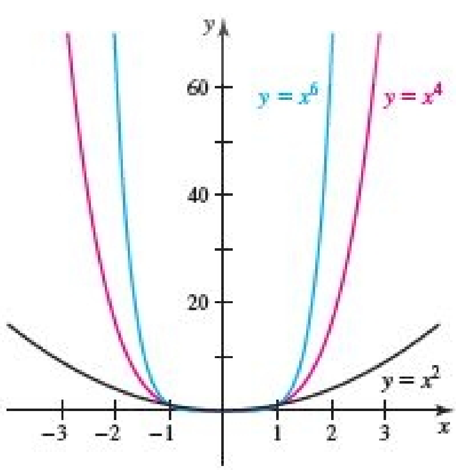
Infinite limits at infinity describe the behavior of all polynomials of degree greater than 0. The simplest examples are provided by functions of the form $f(x) = x^n$ where n is a positive integer. Since positive powers of large numbers are large, this means that for all n,

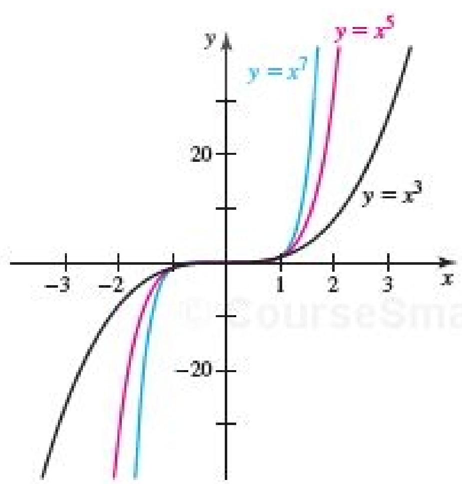
$$\lim_{x\to\infty} x^n = +\infty.$$

Limits at $-\infty$ are only slightly more complicated. Since we are now looking at powers of large magnitude *negative* numbers, the product will be either positive or negative depending on whether n is an *even* or *odd* power. In other words,

$$\lim_{x\to-\infty} x^n = \begin{cases} \infty, & \text{if } n \text{ is even} \\ -\infty, & \text{if } n \text{ is odd} \end{cases}$$

This is illustrated below.





Let's show that when we have any polynomial, its behavior as $x \to \pm\infty$ is completely determined by its highest power. That is if $p(x)$ is a degree n polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then $\lim_{x\to\infty} p(x) = \lim_{x\to\infty} a_n x^n$ and $\lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} a_n x^n$. We can see this by factoring out x^n from $p(x)$

$$\begin{aligned}
\lim_{x\to\infty} p(x) &= \lim_{x\to\infty} a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \\
&= \lim_{x\to\infty} x^n \cdot \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{x^n} \\
&= \lim_{x\to\infty} x^n \left(a_n + \underbrace{\frac{a_{n-1}}{x}}_{\to 0} + \cdots + \underbrace{\frac{a_1}{x^{n-1}}}_{\to 0} + \underbrace{\frac{a_0}{x^n}}_{\to 0} \right) \\
&= \lim_{x\to\infty} a^n x^n.
\end{aligned}$$

It is worth gathering all of these observations together in a theorem, though they should seem quite intuitive or natural.

THEOREM 7.1.2 (Limits of Powers and Polynomials). Let n be a positive integer. Then

(1) If n is even, then $\lim_{x\to\infty} x^n = +\infty$ and $\lim_{x\to-\infty} x^n = +\infty$.

(2) If n is odd, then $\lim_{x\to\infty} x^n = +\infty$ and $\lim_{x\to-\infty} x^n = -\infty$.

(3) (Highest Power) If $p(x)$ is a degree n polynomial, then $\lim_{x\to\infty} p(x) = \lim_{x\to\infty} a_n x^n$ and $\lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} a_n x^n$, where $a_n x^n$ is the **highest degree term**.

Note: A polynomial does not have any horizontal asymptotes.

EXAMPLE 7.1.3 (Polynomial Limits at Infinity). A couple of simple examples illustrate the ideas in the theorem. Let $p(x) = 9x^4 - 2x + 1$ and $q(x) = -4x^5 + 7x^2 + 3$. Then

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} 9x^4 - 2x + 1 \overset{\text{Highest Powers}}{=} \lim_{x\to\infty} 9x^4 = +\infty$$

because the degree is 4 (even) and the leading coefficient is 9 (positive). Similarly

We use "HP" to indicate that we are using the highest degree term to evaluate the limit of the polynomial.

$$\lim_{x\to-\infty} p(x) \overset{\text{HP}}{=} \lim_{x\to-\infty} 9x^4 - 2x + 1 \lim_{x\to-\infty} 9x^4 = +\infty$$

because the degree is still even and the leading coefficient is 9 (positive). Now

$$\lim_{x\to\infty} q(x) \overset{\text{HP}}{=} \lim_{x\to\infty} -4x^5 + 7x^2 + 3 = \lim_{x\to\infty} -4x^5 = -\infty$$

because the degree is 3 (odd) and the leading coefficient is -3 (negative). Similarly

$$\lim_{x\to-\infty} q(x) \overset{\text{HP}}{=} \lim_{x\to-\infty} -4x^5 + 7x^2 + 3 = \lim_{x\to-\infty} -4x^5 = +\infty$$

because the degree is 3 (odd) and the leading coefficient is -3 (negative) and the limit is approaching negative infinity.

These are relatively easy... just think about the sign of the highest degree term as $x \to +\infty$ or $x \to -\infty$.

7.2 *HA's and Rational Functions*

The key to finding horizontal asymptotes for rational functions is to *divide the numerator and denominator by x to the degree (highest power of x) of the denominator*.

EXAMPLE 7.2.1. Determine the HA's of $f(x) = \frac{4x^4 - 2x^3 + 7}{3x^4 + 2x^2 + 1}$.

Solution. We need to determine the limits as $x \to \pm\infty$. Dividing by x^4 (the degree of the denominator is 4)

$$\lim_{x\to\infty} \frac{4x^4 - 2x^3 + 7}{3x^4 + 2x^2 + 1} = \lim_{x\to\infty} \frac{4 - \frac{2}{x} + \frac{7}{x^4}}{3 + \frac{2}{x^2} + \frac{1}{x^4}} = \frac{4 - 0 + 0}{3 + 0 + 0} = \frac{4}{3}.$$

So $y = \frac{4}{3}$ is an HA. What about as $x \to -\infty$?

$$\lim_{x\to-\infty} \frac{4x^4 - 2x^3 + 7}{3x^4 + 2x^2 + 1} = \lim_{x\to-\infty} \frac{4 - \frac{2}{x} + \frac{7}{x^4}}{3 + \frac{2}{x^2} + \frac{1}{x^4}} = \frac{4}{3}.$$

EXAMPLE 7.2.2. Determine $\lim_{x\to-\infty} \frac{3x^2-2x}{9x^3+1}$.

Solution. Dividing by x^3 (the degree of the denominator is 3)

$$\lim_{x\to-\infty} \frac{3x^2 - 2x}{9x^3 + 1} = \lim_{x\to-\infty} \frac{\frac{3}{x} - \frac{2}{x^2}}{9 + \frac{1}{x^3}} = \frac{0 - 0}{9 + 0} = 0.$$

So $y = 0$ is an HA.

EXAMPLE 7.2.3. Determine $\lim_{x\to-\infty} \frac{4x^3-2}{2x^2+x}$.

Solution. Dividing by x^2 (the degree of the denominator is 2)

$$\lim_{x\to-\infty} \frac{4x^3 - 2}{2x^2 + x} = \lim_{x\to-\infty} \frac{4x - \frac{2}{x^2}}{2 + \frac{1}{x}} = \lim_{x\to-\infty} \frac{4x}{2} = -\infty.$$

There is no HA.

EXAMPLE 7.2.4. Determine $\lim_{x\to\infty} \frac{2\sqrt{x}}{2x+1}$.

Solution. Dividing by x (the degree of the denominator is 1)

$$\lim_{x\to\infty} \frac{2x^{1/2}}{2x + 1} = \lim_{x\to\infty} \frac{\frac{2}{x^{1/2}}}{2 + \frac{1}{x}} = \frac{0}{2 + 0} = 0.$$

There is an HA at $y = 0$.

Dominant Powers

Here are a couple of quick observations we can make about a rational function $\frac{p(x)}{q(x)}$ from these examples. The limits depend on the highest degree terms in both the numerator and denominator. So we can focus on just those terms. Return to Example 7.2.2. Ignoring the lower degree terms in the numerator and denominator we get

$$\lim_{x\to-\infty} \frac{3x^2 - 2x}{9x^3 + 1} \overset{\text{Highest Powers}}{=} \lim_{x\to-\infty} \frac{3x^2}{9x^3} = \lim_{x\to-\infty} \frac{\frac{3}{x}}{9} = \frac{0}{9} = 0,$$

as before.

EXAMPLE 7.2.5. Here are a few more examples. Indicate this process by using HP over the equals sign when you employ it.

$$\lim_{x\to-\infty} \frac{3x^2 - 2x + 1}{2x^4 + 6x} \overset{\text{HP}}{=} \lim_{x\to-\infty} \frac{3x^2}{2x^4} = \lim_{x\to-\infty} \frac{\frac{3}{x^2}}{2} = \frac{0}{2} = 0.$$

Similarly

$$\lim_{x\to\infty} \frac{2x^3 + 2x^2 + 1}{5x^3 + x} \overset{\text{HP}}{=} \lim_{x\to\infty} \frac{2x^3}{5x^3} = \frac{2}{5}.$$

This method works with fractional powers

$$\lim_{x\to\infty} \frac{2x^{3/2} + x - 1}{5x + 7} \overset{\text{HP}}{=} \lim_{x\to\infty} \frac{2x^{3/2}}{5x} = \lim_{x\to\infty} \frac{2x^{1/2}}{5} = +\infty.$$

More examples with Square Roots

Remember that

$$\sqrt{x^2} = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

Using highest powers

$$\lim_{x\to+\infty} \frac{\sqrt{x^2+7x+1}}{2x+2} \overset{\text{HP}}{=} \lim_{x\to+\infty} \frac{\sqrt{x^2}}{2x} = \lim_{x\to+\infty} \frac{|x|}{2x} \overset{x\geq 0}{=} \lim_{x\to+\infty} \frac{x}{2x} = \frac{1}{2}.$$

Now compare what happens with

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+7x+1}}{2x+2} \overset{\text{HP}}{=} \lim_{x\to-\infty} \frac{\sqrt{x^2}}{2x} = \lim_{x\to-\infty} \frac{|x|}{2x} \overset{x\leq 0}{=} \lim_{x\to-\infty} \frac{-x}{2x} = -\frac{1}{2}.$$

This is why it is important to calculate the limits for both $x \to +\infty$ and $x \to -\infty$. So there are two HA's for this function: $y = \pm\frac{1}{2}$.

EXAMPLE 7.2.6. Determine the HA's for

$$f(x) = \frac{\sqrt{16x^2+5x}}{3x+1}.$$

Solution. Using highest powers

$$\lim_{x\to+\infty} \frac{\sqrt{16x^2+5x}}{3x+1} \overset{\text{HP}}{=} \lim_{x\to+\infty} \frac{\sqrt{16x^2}}{3x} = \lim_{x\to+\infty} \frac{|4x|}{3x} \overset{x\geq 0}{=} \lim_{x\to+\infty} \frac{4x}{3x} = \frac{4}{3}.$$

Now compare what happens with

$$\lim_{x\to-\infty} \frac{\sqrt{16x^2+5x}}{3x+1} \overset{\text{HP}}{=} \lim_{x\to-\infty} \frac{\sqrt{16x^2}}{3x} = \lim_{x\to-\infty} \frac{|4x|}{3x} \overset{x\leq 0}{=} \lim_{x\to-\infty} \frac{-4x}{3x} = -\frac{4}{3}$$

because in this case $x < 0$ so $|4x| = -4x$. So $y = \pm\frac{4}{3}$ are both HA's.

EXAMPLE 7.2.7. Determine $\lim_{x\to-\infty} \frac{\sqrt[3]{x^2+2x}}{x+1}$. This time

$$\lim_{x\to-\infty} \frac{\sqrt[3]{x^2+2x}}{x+1} \overset{\text{HP}}{=} \lim_{x\to-\infty} \frac{\sqrt[3]{x^2}}{x} = \lim_{x\to-\infty} \frac{x^{2/3}}{x} = \lim_{x\to-\infty} \frac{1}{x^{1/3}} = 0$$

Your text states the following theorem which we include for completeness. But rather than memorize this this theorem, just employ the dominant or highest powers technique.

THEOREM 7.2.8 (Limits at Infinity for Rational Functions). Let $p(x)$ and $q(x)$ be polynomials.

(1) If the degree of the numerator is *less* than the degree of the denominator, then $\lim_{x\to\pm\infty} \frac{p(x)}{q(x)} = 0$ and $x = 0$ is a HA.

(2) If the degree of the numerator is *the same* as the degree of the denominator, then $\lim_{x\to\pm\infty} \frac{p(x)}{q(x)} = \frac{a}{b}$ where a and b are the leading coefficients of p and q and $x = \frac{a}{b}$ is a HA.

(3) If the degree of the numerator is *larger* than the degree of the denominator, then $\lim_{x\to\pm\infty} \frac{p(x)}{q(x)} = \infty$ or $-\infty$ depending on the highest powers and their coefficients in the polynomials $p(x)$ and $q(x)$. There is no HA.

EXAMPLE 7.2.9. Now we will use these observations to evaluate the following rational functions

$$\lim_{t\to+\infty} \frac{3t-1}{t^3-1} = 0$$

because the degree of the numerator is 1 and the denominator is 3. Next

$$\lim_{x\to+\infty} \frac{2x^3+x^2+1}{4x^3-2} = \frac{2}{4} = \frac{1}{2}$$

because the degrees of the numerator and denominator are equal.

$$\lim_{x\to-\infty} \frac{5x^5-x}{10x^2+1} = \lim_{x\to-\infty} \frac{5x^5}{10x^2} = \lim_{x\to-\infty} 2x^3 = -\infty.$$

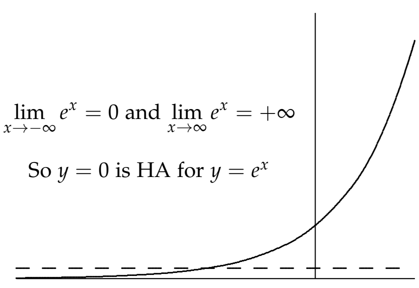
Here the degree of the numerator is larger than the degree of the denominator.

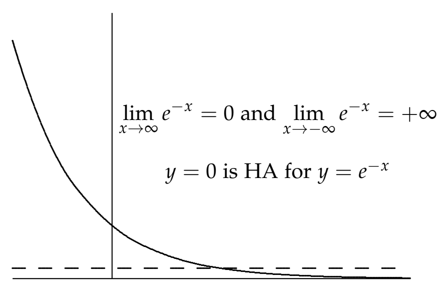
The End Behavior of the Natural Log and Exponential Functions

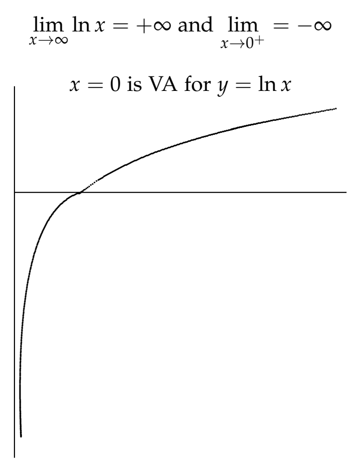
Earlier we examined the graphs of $y = e^x$ and $y = e^{-x}$. Using these graphs and the the graph of $y = \ln x$ we make some final observations.

THEOREM 7.2.10 (The End Behavior of the Natural Log and Exponential Functions). The end behavior of e^x and e^{-x} on $9-\infty,\infty)$ and $\ln x$ on $(0,+\infty)$ is given by

$$\begin{aligned} \lim_{x\to\infty} e^x &= +\infty \quad \text{and} \quad \lim_{x\to-\infty} e^x = 0 \\ \lim_{x\to\infty} e^{-x} &= 0 \quad \text{and} \quad \lim_{x\to-\infty} e^{-x} = +\infty \\ \lim_{x\to0^+} \ln x &= -\infty \quad \text{and} \quad \lim_{x\to\infty} \ln x = +\infty \end{aligned}$$







Other Limits at Infinity

Using the basic limit laws and the few limit facts we have established, we can determine other limits at infinity. For example, since $\lim_{x\to\infty} e^{-x} = 0$, then

$$\lim_{x\to\infty} \frac{4}{3e^x} = \lim_{x\to\infty} \frac{4}{3}e^{-x} = \frac{4}{3}(0) = 0.$$

Or using the sum rule for limits,

$$\lim_{x\to\infty} \frac{x^2+1}{3x^2} - \frac{10}{x} = \lim_{x\to\infty} \frac{x^2+1}{3x^2} - \lim_{x\to\infty} \frac{10}{x} = \frac{1}{3} - 0 = \frac{1}{3}.$$