

## 5-Minute Review: Absolute Values

Absolute values are important to help us carefully define limits. Recall from Lab 1:

**DEFINITION 10.1.1.** If  $a$  is positive, then  $|x| < a$  means  $-a < x < a$ .

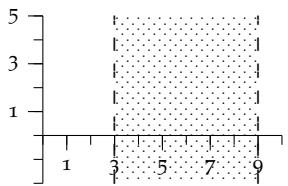
**EXAMPLE 10.1.2.** Find all the points in the plane that simultaneously satisfy both  $|x - 6| < 3$  and  $|y - 2| < 1$ . Draw a graph of the solution set.

**SOLUTION.** Using the definition above,

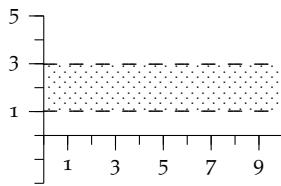
$$|x - 6| < 3 \text{ means } -3 < x - 6 < 3, \text{ so } 3 < x < 9.$$

The set of such points in the *plane* is a vertical strip between the lines  $x = 3$  and  $x = 9$ .

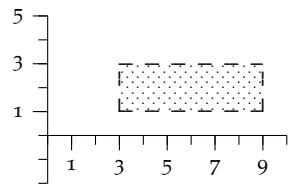
See figure (a) below.



(a):  $|x - 6| < 3$



(b):  $|y - 2| < 1$



(c):  $|x - 6| < 3$  and  $|y - 2| < 1$

Next,

$$|y - 2| < 1 \text{ means } -1 < y - 2 < 1, \text{ so } 1 < y < 3.$$

The set of such points in the *plane* is a horizontal strip between the lines  $y = 1$  and  $y = 3$ . See figure (b) above.

Finally, to satisfy both inequalities simultaneously, we use the intersection of the sets in (a) and (c) that produces the rectangular set in figure (c) above.

**YOU TRY IT 10.1.** Solve each of the following using Definition 10.1.1.

(a)  $|x - 3| = 4$     (b)  $|x - 5| < 2$     (c)  $|y - 2| < 1$

(d) Graph (below, left) all points in the plane that satisfy *both* (b) and (c).

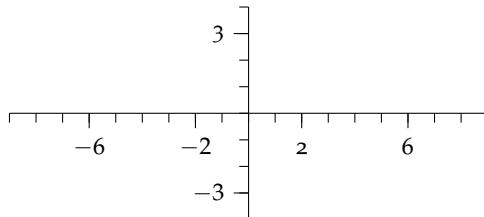
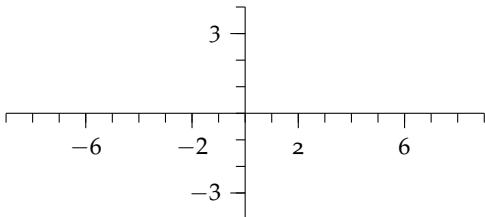
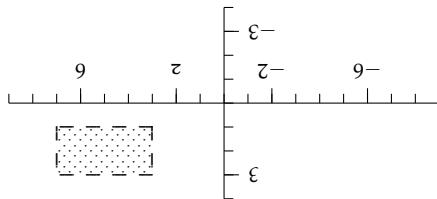
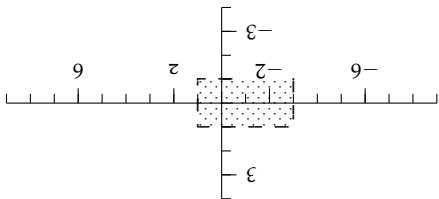


Figure 10.1: Graph your answers to part (d) on the left and part (e) on the right.

(e) What shape figure did you get for your solution to part (d)? Did the solution include the boundary of the figure? Why?

(f) Find and draw (above, right) the solution set to  $|x + 1| < 2$  and  $|y| < 1$ .



(d) We get a rectangle. No, the boundary is not included because both inequalities were strict.

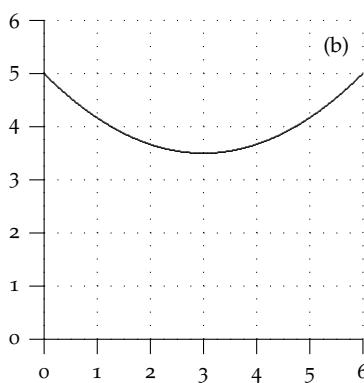
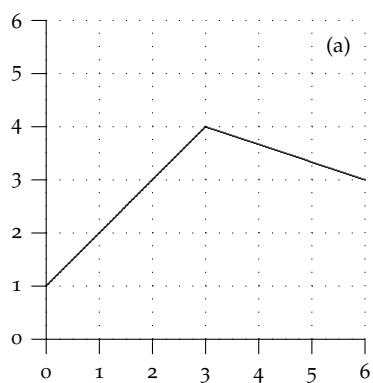
(e)  $|y - 2| < 1$  means  $-1 < y - 2 < 1$  or  $1 < y < 3$ .

(f)  $|x - 5| < 2$  means  $-2 < x - 5 < 2$  or  $3 < x < 7$ .

(g)  $|x - 3| = 4$  means  $x$  and 3 are 4 apart, so  $x = 7$  or  $-1$ .

ANSWER TO YOU TRY IT 10.1.

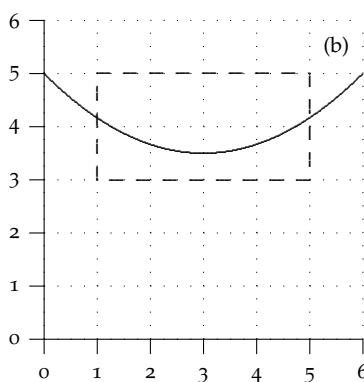
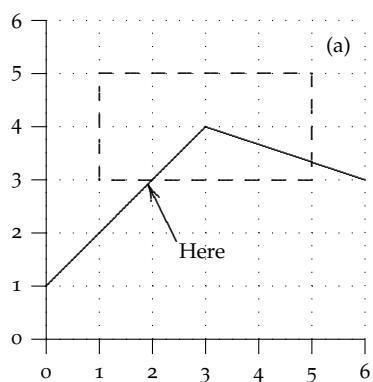
**EXAMPLE 10.1.3.** In You Try It 10.1 we saw that all the points in the plane that satisfy  $|x - 5| < 2$  and  $|y - 2| < 1$  simultaneously form a rectangle (that does not include its perimeter). We now apply this idea to graphs of functions. Suppose that  $y = f(x)$  is the graph shown below.



Determine which of the graphs of  $y = f(x)$  below satisfy the following condition:

If  $|x - 3| < 2$ , then  $|f(x) - 4| < 1$ .

**SOLUTION.** First: Make sure you understand what this means. Whenever the first inequality is true, the second inequality must also be true. Remember that  $f(x)$  is the  $y$ -coordinate. As in the previous example, the two inequalities  $|x - 3| < 2$  and  $|f(x) - 4| < 1$  determine a rectangle in the plane, as shown below, where if  $1 < x < 5$ , then we should have  $3 < f(x) < 5$  for each graph.



In (a) for the function  $f(x)$  to satisfy the condition requires that when  $x$  is between 1 and 3 then  $3 < f(x) < 5$ . In other words, when  $x$  is still between 1 and 3 the graph of  $f$  must stay *inside* the rectangle. Notice in the situation above, the graph exits the rectangle (where it says 'Here') while  $x$  is between 1 and 3, so the condition is not satisfied. For example,  $f(1.5) = 2.5$  lies outside the rectangle.

In (b), when  $x$  is between 1 and 3 the graph of  $f$  always lies *inside* the rectangle. So the condition *is* satisfied for the second graph.

## The Limit Definition

We are ready to make precise what we mean the term 'limit.' So far we have worked with an intuitive understanding of the term:

**DEFINITION 10.1.4 (Informal Definition of Limit).** We write  $\lim_{x \rightarrow a} g(x) = L$  and say that the **limit of  $g(x)$  as  $x$  approaches  $a$**  if we can make  $g(x)$  arbitrarily close to  $L$  by taking  $x$  sufficiently close to (but not equal to)  $a$ .

One of the points that distinguishes mathematics from other disciplines is the precision of its definitions. So compare the definition above to the

**DEFINITION 10.1.5 (Formal Definition of Limit).** Let  $f$  be a function defined on some open interval containing  $a$ , except perhaps at  $a$  itself. We say that  $\lim_{x \rightarrow a} f(x) = L$  if for every number  $\varepsilon > 0$  there is a corresponding number  $\delta > 0$  so that

$$\text{if } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \varepsilon.$$

Each inequality makes precise some aspect of the informal definition of limit.

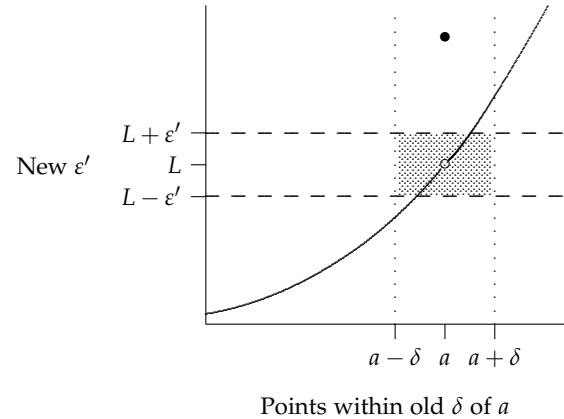
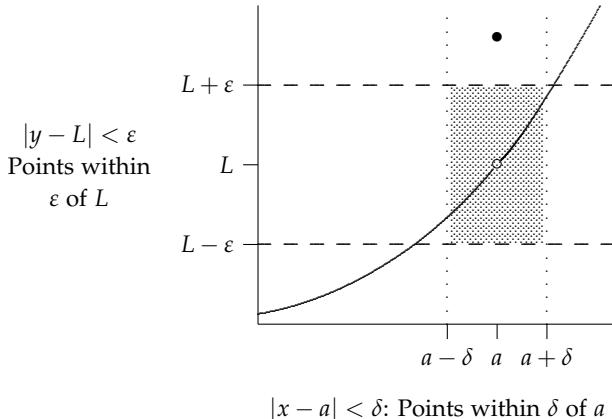
- $|f(x) - L| < \varepsilon$  means "we can make  $f(x)$  arbitrarily close to  $L$ " (within  $\varepsilon$  of  $L$ )
- by taking  $|x - a| < \delta$ , i.e., "taking  $x$  sufficiently close to  $a$ " (within  $\delta$  of  $a$ )
- The inequality  $0 < |x - a|$  simply means " $x$  is not equal to  $a$ ."

**EXERCISE 10.1.6.** Turn the points above into inequalities:

- $|f(x) - L| < \varepsilon$  becomes
- $|x - a| < \delta$  becomes
- $0 < |x - a|$  becomes

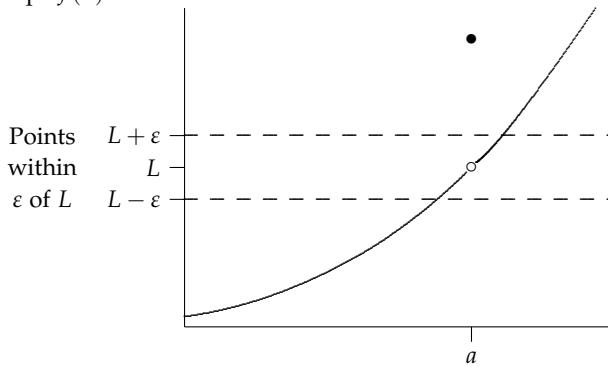
## Examples of Using the Limit Definition

**EXAMPLE 10.1.7.** On the left: For the given  $\varepsilon$ , the selected  $\delta$  keeps  $f(x)$  within the horizontal band, that is, within  $\varepsilon$  of  $L$  over the interval from  $a - \delta$  to  $a + \delta$  (except perhaps at  $a$ ). On the right: With a smaller  $\varepsilon'$ , the old  $\delta$  may fail. Is it possible to find a new  $\delta$  that works?



On the left: When  $x$  is trapped between  $a - \delta$  and  $a + \delta$ , then  $f(x)$  is trapped between  $L - \varepsilon$  and  $L + \varepsilon$  EXCEPT at  $x = a$ .

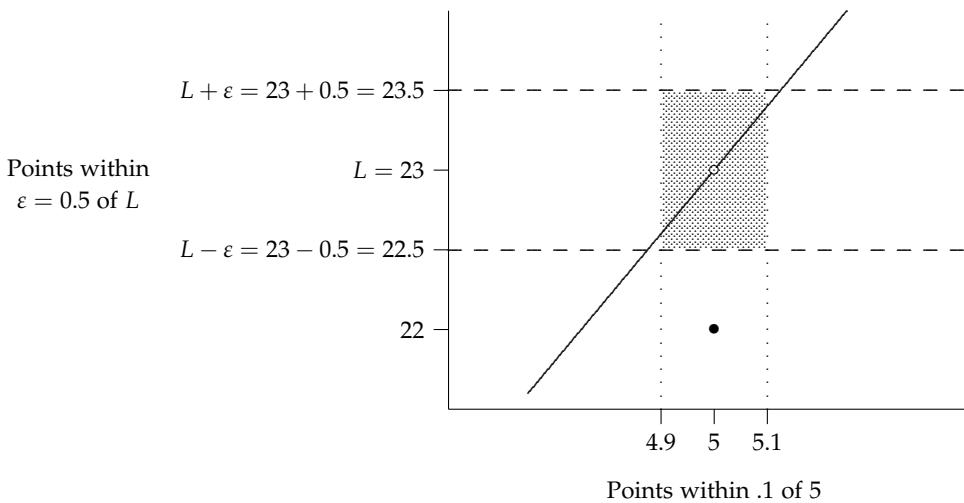
**EXAMPLE 10.1.8.** The formal definition says that for each and every  $\varepsilon$ , we need to be able to find a  $\delta$ . With the same function as before, for a smaller choice of  $\varepsilon$ , the earlier value of  $\delta$  might not work. However: As long as we can find a new  $\delta$  for each new  $\varepsilon$ , the limit will exist. **For the smaller choice of  $\varepsilon$ , draw a  $\delta$  interval about  $a$  that satisfies the limit definition**—that traps  $f(x)$  within  $\varepsilon$  of  $L$ .



To repeat: As long as we can find a new  $\delta$  for each new  $\varepsilon$ , the limit will exist.

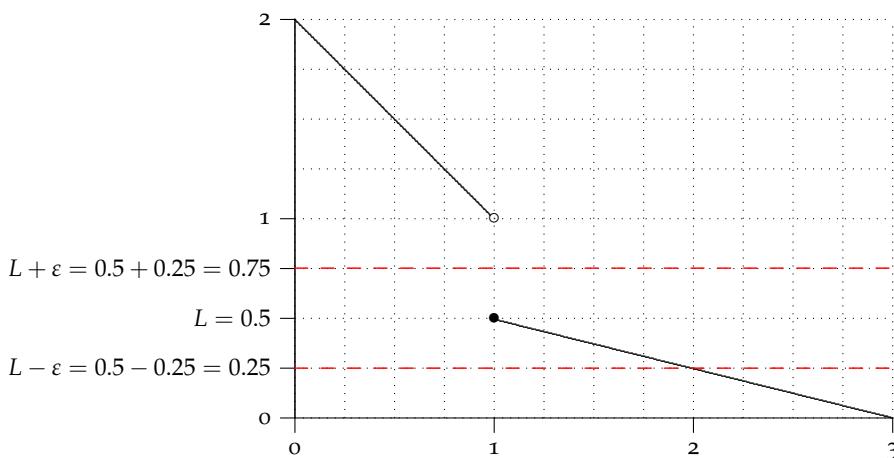
**EXAMPLE 10.1.9.** For the function  $f(x) = \begin{cases} 4x + 3, & \text{if } x \neq 5 \\ 22, & \text{if } x = 5 \end{cases}$ .

Use absolute values to describe the shaded region.



**EXAMPLE 10.1.10.** Let  $f(x) = \begin{cases} 2 - x, & \text{if } x < 1 \\ 1 - \frac{1}{2}x, & \text{if } x \geq 1 \end{cases}$ . Intuitively we can see that  $\lim_{x \rightarrow 1} f(x)$

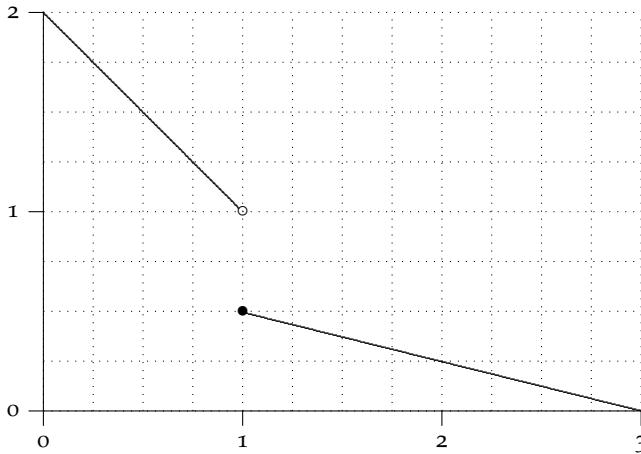
DNE. For example we show that  $\lim_{x \rightarrow 1} f(x) \neq 0.5$  by using  $\varepsilon = 0.25$ . Now there is NO  $\delta > 0$  that satisfies the limit definition. That is, for every  $\delta > 0$ , if  $0 < |x - 1| < \delta$ , then  $|f(x) - 0.5| > 0.25 = \varepsilon$ . There are always points outside (above) the red ' $\varepsilon$ '-corridor no matter how small  $\delta$  is.



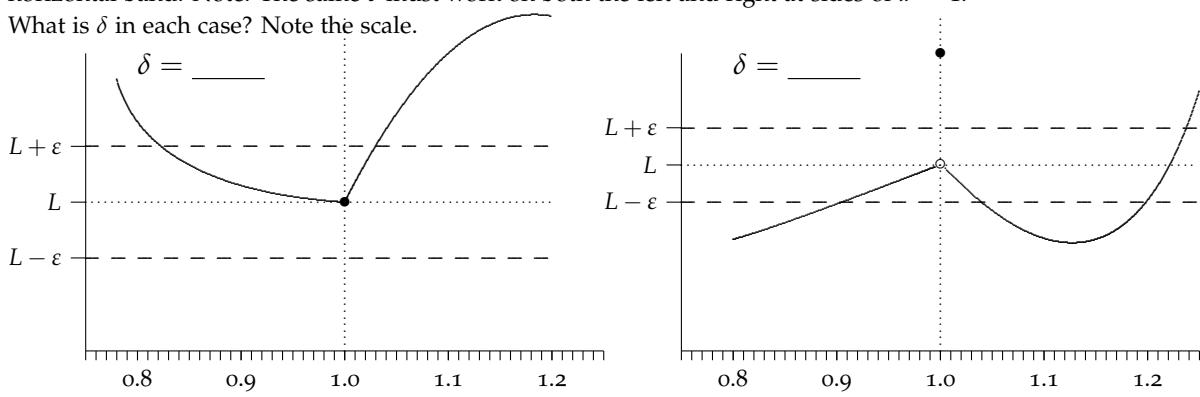
There is no  $\delta$  that will trap  $f(x)$  within 0.25 of 0.5. There is no  $\delta$  so that if  $0 < |x - 1| < \delta$ , then  $|f(x) - 0.5| < 0.25 = \varepsilon'$ .

**YOU TRY IT 10.2.** Here's another copy of the graph of the function in Example 10.1.10.

Find a particular value of  $\epsilon$  to show that  $\lim_{x \rightarrow 1} f(x) \neq 1$ .

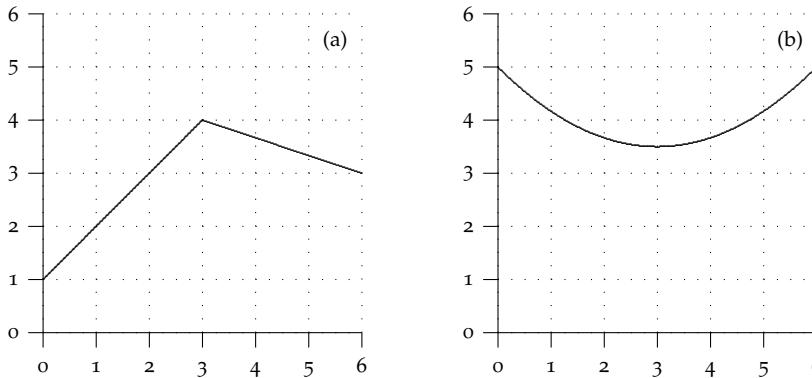


**YOU TRY IT 10.3.** In each figure below, for the given choice of  $\epsilon$ , find and draw a  $\delta$  interval (a vertical strip) about  $a = 1$  which satisfies the limit definition—that traps the function in the horizontal band. Note: The same  $\delta$  must work on both the left and right at sides of  $a = 1$ . What is  $\delta$  in each case? Note the scale.



**EXAMPLE 10.1.11.** We know how to solve inequalities involving absolute values. Now apply this to functions. Determine whether the following statement is true or false for each of the functions graphed below.

If  $|x - 3| < 2$ , then  $|f(x) - 4| < 1$ .



**SOLUTION.** Convert the absolute values to ordinary inequalities and draw the strips.

- Do this now for  $|x - 3| < 2$  and  $|f(x) - 4| < 1$  and draw the corresponding strips on the graphs above. Remember  $f(x)$  represents the  $y$ -coordinate
- For the statement, "If  $|x - 3| < 2$ , then  $|f(x) - 4| < 1$ ." to be true, means when  $x$  is in the vertical strip, then  $f(x)$  must be in the horizontal strip. In other words, the graph of the function must be inside the rectangle when  $|x - 3| < 2$ . Do both functions satisfy this condition?

## Limit Proofs: A Two-Step Process

This section demonstrates how mathematicians use the formal definition of limit to give careful proofs of limit calculations. We will look at some very simple examples that illustrate this idea.<sup>1</sup> We will use a two-stage process to prove that  $\lim_{x \rightarrow a} f(x) = L$ . Remember that

$\lim_{x \rightarrow a} f(x) = L$  if for every number  $\varepsilon > 0$  there is a corresponding number  $\delta > 0$  so that

$$\text{if } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \varepsilon.$$

So we need to

1. **SCRAP WORK:** Find  $\delta$ . We do this by letting  $\varepsilon$  be an arbitrary positive number and then we use the inequality  $|f(x) - L| < \varepsilon$  to work ‘backwards’ to a statement of the form  $|x - a| < \delta$ , where  $\delta$  depends only on  $\varepsilon$ .
2. **ARGUMENT:** Write the proof. For any  $\varepsilon > 0$ , assume  $0 < |x - a| < \delta$  and use the work in Step 1 to prove that  $|f(x) - L| < \varepsilon$ . (Here we work ‘forward.’)

The first step is essentially ‘scrap work’ for the proof. The second step is the actual ‘proof’ that our choice of  $\delta$  works. This is clearer in an example.

**EXAMPLE 10.1.12** (The Limit of a Linear function). Prove that  $\lim_{x \rightarrow 2} (3x + 5) = 11$  using the formal definition of limit.

**SOLUTION.** **SCRAP WORK:** Find  $\delta$ . In this case  $a = 2$  and  $L = 11$ . Assume that  $\varepsilon > 0$  is given (but arbitrary). Work backwards:

Translate from the general to this particular function.

$$|f(x) - L| < \varepsilon \xrightarrow{\text{Translate}} |(3x + 5) - 11| < \varepsilon$$

Now simplify the absolute value.

$$\xrightarrow{\text{Simplify}} |3x - 6| < \varepsilon$$

Factor out the constant in front of  $x$ .

$$\xrightarrow{\text{Factor}} 3|x - 2| < \varepsilon$$

Solve for  $|x - a|$ .

$$\xrightarrow{\text{Solve}} |x - 2| < \frac{\varepsilon}{3}.$$

We now have  $|x - a| < \delta$  where  $a = 2$  and  $\delta = \frac{\varepsilon}{3}$ .

$$\therefore \text{Choose } \delta = \frac{\varepsilon}{3}.$$

At this last step we have an inequality of the form  $|x - a| < \delta$ . We identify  $\delta$  as  $\frac{\varepsilon}{3}$ . Notice how  $\delta$  depends on  $\varepsilon$ . In particular, as  $\varepsilon$  gets smaller, so does  $\delta$ . We saw this geometrically in the graphs in the first part of this handout. Now we are ready to write the actual proof.

**ARGUMENT:** Let  $\varepsilon > 0$  be given. Choose  $\delta = \frac{\varepsilon}{3}$ . If  $0 < |x - 2| < \delta = \frac{\varepsilon}{3}$ , then

$$\begin{aligned} |f(x) - L| &= |(3x + 5) - 11| \xrightarrow{\text{Simplify}} |3x - 6| \\ &\xrightarrow{\text{Factor}} 3|x - 2| \end{aligned}$$

Because we know  $|x - 2| < \frac{\varepsilon}{3}$ , we can substitute and make an inequality

$$|x - 2| < \frac{\varepsilon}{3} \quad 3 \cdot \frac{\varepsilon}{3} = \varepsilon.$$

The proof is complete: We have shown that for any  $\varepsilon > 0$ , if  $0 < |x - 2| < \delta = \frac{\varepsilon}{3}$ , then  $|(3x + 5) - 11| < \varepsilon$ . Having done the necessary scrap work, the entire proof consists of three short sentences.

<sup>1</sup> To learn about more complex limit calculations, take Math 331.

**YOU TRY IT 10.4.** Once we have found  $\delta$  in terms of  $\varepsilon$ , no matter what  $\varepsilon$  we choose, an appropriate  $\delta$  can be found. In Example 10.1.12, if  $\varepsilon = 0.01$ , what would  $\delta$  be? Or if  $\varepsilon = 0.0001$ , how would you choose  $\delta$ ?

Also if  $\delta = \frac{\varepsilon}{3}$  ‘works,’ then any smaller positive value of  $\delta$  will also work. Show that  $\delta = \frac{\varepsilon}{4}$  ‘works’ in the proof above.

**EXAMPLE 10.1.13 (Another Linear function).** Prove that  $\lim_{x \rightarrow 1} (2 - 5x) = -3$  using the formal definition of limit.

**SOLUTION.** **SCRAP WORK:** Find  $\delta$ . In this case  $a = 1$  and  $L = -3$ . Assume that  $\varepsilon > 0$  is given (but arbitrary). Work backwards:

$$\begin{aligned} |f(x) - L| &< \varepsilon \xrightarrow{\text{Translate}} |(2 - 5x) - (-3)| < \varepsilon \\ &\xrightarrow{\text{Simplify}} |-5x + 5| < \varepsilon \end{aligned}$$

Be careful to factor out  $|-5| = 5$ , not just  $-5$ .

$$\begin{aligned} &\xrightarrow{\text{Factor}} |-5| \cdot |x - 1| < \varepsilon \\ &\xrightarrow{\text{Solve}} |x - 1| < \frac{\varepsilon}{|-5|} = \frac{\varepsilon}{5}. \quad \therefore \text{Choose } \delta = \frac{\varepsilon}{5}. \end{aligned}$$

We have our inequality of the form  $|x - a| < \delta$  with  $\delta = \frac{\varepsilon}{5}$ . Now write the proof.

**ARGUMENT:** Let  $\varepsilon > 0$  be given. Choose  $\delta = \frac{\varepsilon}{5}$ . If  $0 < |x - 1| < \delta = \frac{\varepsilon}{5}$ , then

$$\begin{aligned} |f(x) - L| &= |(2 - 5x) - (-3)| \\ &\stackrel{\text{Simplify}}{=} |-5x + 5| \\ &\stackrel{\text{Factor}}{=} |-5| \cdot |x - 1| \end{aligned}$$

Because we know  $|x - 1| < \frac{\varepsilon}{5}$ , we can substitute and make an inequality

$$|x - 1| < \frac{\varepsilon}{5} \quad 5 \cdot \frac{\varepsilon}{5} = \varepsilon. \quad \text{Done!}$$

The proof is complete: We have shown that for any  $\varepsilon > 0$ , if  $0 < |x - 1| < \delta = \frac{\varepsilon}{5}$ , then  $|(2 - 5x) - (-3)| < \varepsilon$ .

**EXAMPLE 10.1.14 (One More).** Prove that  $\lim_{x \rightarrow -3} (10x + 8) = -22$  using the formal definition of limit.

**SOLUTION.** **SCRAP WORK:** Find  $\delta$ . In this case  $a = -3$  and  $L = -22$ . Be careful of all the negatives. Assume that  $\varepsilon > 0$  is given. Then

$$\begin{aligned} |(10x + 8) - (-22)| &< \varepsilon \xrightarrow{\text{Simplify}} |10x + 30| < \varepsilon \\ &\xrightarrow{\text{Factor}} 10|x + 3| < \varepsilon \\ &\xrightarrow{\text{Rewrite}} 10|x - (-3)| < \varepsilon \\ &\xrightarrow{\text{Solve}} |x - (-3)| < \frac{\varepsilon}{10}. \quad \therefore \text{Choose } \delta = \frac{\varepsilon}{10}. \end{aligned}$$

Notice that we wrote  $|x + 3|$  as  $|x - (-3)|$  so that our inequality would have the correct form:  $|x - a| < \delta$  with  $\delta = \frac{\varepsilon}{10}$ . Do the proof.

**ARGUMENT:** Given  $\varepsilon > 0$ . Assume that  $0 < |x - (-3)| < \delta = \frac{\varepsilon}{10}$ . Then

$$|(10x + 8) - (-22)| = |10x + 30| = 10|x + 3| = 10|x - (-3)| \xrightarrow{|x - (-3)| < \frac{\varepsilon}{10}} 10 \cdot \frac{\varepsilon}{10} = \varepsilon.$$

Short and sweet!

Answers to **YOU TRY IT 10.4**:  $\delta = \frac{\varepsilon}{3} = \frac{0.01}{3}$ , and  $\delta = \frac{0.0001}{3}$ .  
This time  $|3x - 6| = 3|x - 2| < 3 \cdot \frac{\varepsilon}{4} < \varepsilon$ .

**EXAMPLE 10.1.15.** (Something different) Prove that  $\lim_{x \rightarrow 3} |18 - 6x| = 0$  using the formal definition of limit.

**SOLUTION.** **SCRAP WORK:** Find  $\delta$ . In this case  $a = 3$  and  $L = 0$ . Assume that  $\varepsilon > 0$  is given. Then

$$\begin{aligned} ||18 - 6x| - 0| &< \varepsilon \stackrel{\text{Simplify}}{\iff} |18 - 6x| < \varepsilon \\ &\stackrel{\text{Factor}}{\iff} |-6||x - 3| < \varepsilon \stackrel{\text{Solve}}{\iff} |x - 3| < \frac{\varepsilon}{6}. \end{aligned}$$

So  $\delta = \frac{\varepsilon}{6}$ . Do the proof.

**ARGUMENT:** Given  $\varepsilon > 0$ . Choose  $\delta = \frac{\varepsilon}{6}$ . If  $0 < |x - 3| < \frac{\varepsilon}{6}$ , then

$$||18 - 6x| - 0| = |18 - 6x| = 10|x + 3| = |-6||x - 3| < 6 \cdot \frac{\varepsilon}{6} = \varepsilon.$$

*What your work should look like.* Here's what I expect that your work will look like when you turn it in.

**EXERCISE 10.1.16.** Prove that  $\lim_{x \rightarrow 5} | -2x + 6 | = -4$  using the formal definition of limit.

Scrap: Given  $\varepsilon > 0$ . Want to find  $\delta > 0$ .

Work backwards

$$\begin{aligned} |f(x) - L| &< \varepsilon \stackrel{\text{Translate}}{\iff} |(-2x+6) - (-4)| < \varepsilon \\ &\iff |-2x + 10| < \varepsilon \\ &\iff |-2| |x - 5| < \varepsilon \\ &\iff |x - 5| < \frac{\varepsilon}{|-2|} = \frac{\varepsilon}{2} \quad ) \\ \text{choose } \delta &= \frac{\varepsilon}{2} \end{aligned}$$

Proof: Given  $\varepsilon > 0$ . If  $0 < |x - 5| < \delta = \frac{\varepsilon}{2}$ ,

$$\begin{aligned} \text{then } |f(x) - L| &= |(-2x+6) - (-4)| \\ &= |-2x + 10| \\ &= |-2| |x - 5| \\ &= 2 |x - 5| \quad \text{we know } |x - 5| < \frac{\varepsilon}{2} \\ &< 2 \cdot \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

**YOU TRY IT 10.5.** Use the formal definition to prove each of the following.

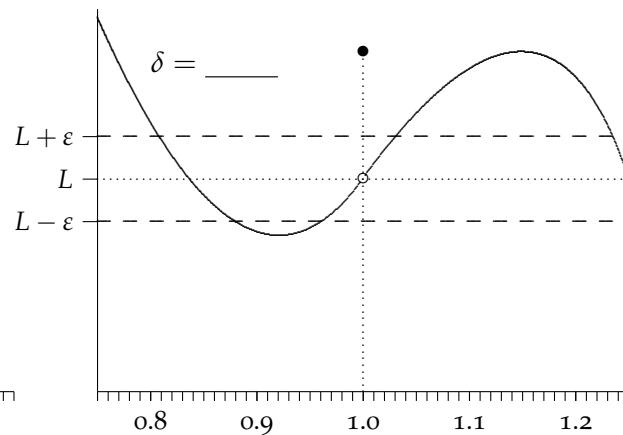
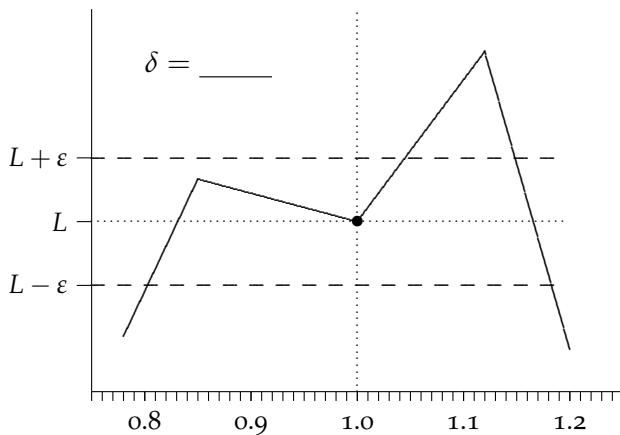
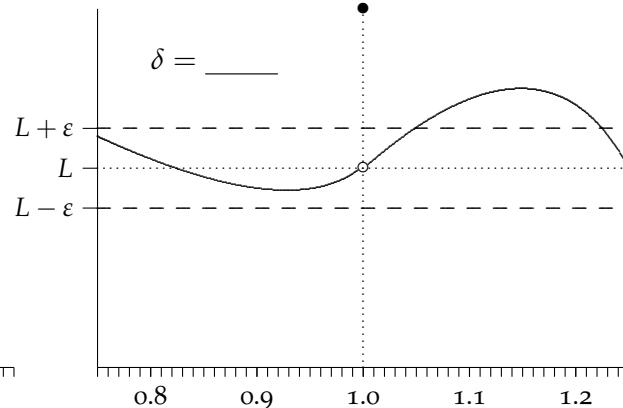
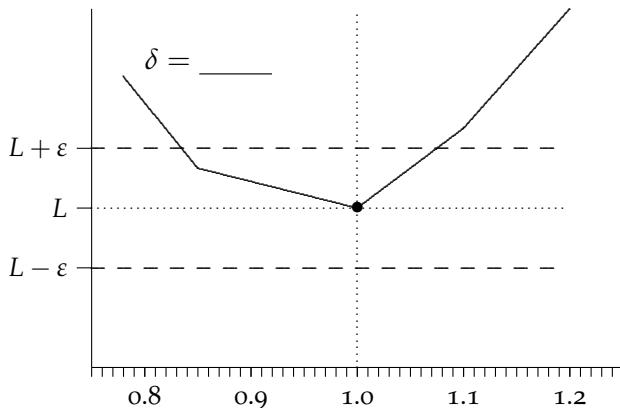
$$(a) \lim_{x \rightarrow 5} (4x + 7) = 27 \quad (b) \lim_{x \rightarrow -4} (-2x - 17) = -9 \quad (c) \lim_{x \rightarrow 12} \left( \frac{x}{2} - 11 \right) = -5$$

The Stuff You Need to Turn In Next Class. Name: \_\_\_\_\_

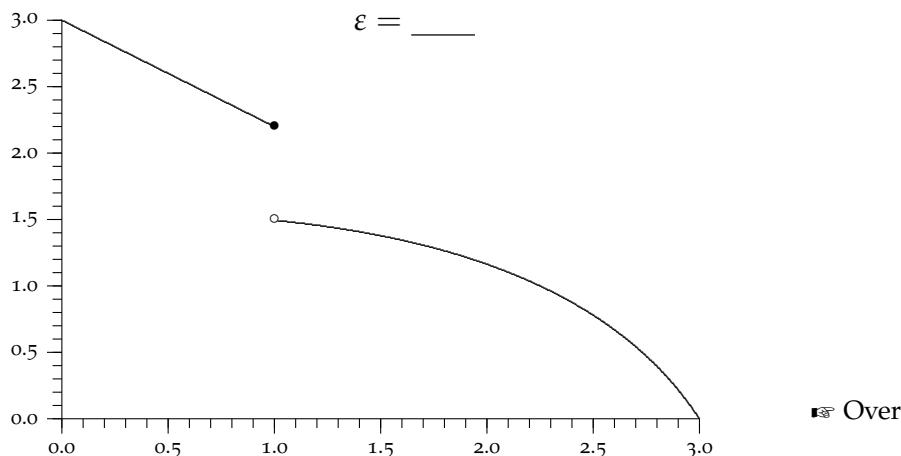
You should work with one partner if at all possible. If you do, hand in one sheet for both of you.

1. In each figure, for the given choice of  $\varepsilon$ , find and draw a  $\delta$  interval (a vertical strip) about  $a = 1$  which satisfies the limit definition. What is  $\delta$  in each case?

Note the scale. Note: In each figure, the same  $\delta$  must work on both the left and right at sides of  $a = 1$ .



2. For the function  $f(x)$  below, show that that  $1.5$  is not  $\lim_{x \rightarrow 1} f(x)$ . To do this, find and draw a horizontal  $\varepsilon$  interval about  $y = 1.5$  for which there is no value of  $\delta$  that will satisfy the limit definition. (For your value of  $\varepsilon$ , you can never trap  $f(x)$  in the corresponding horizontal band.) What is the value of your  $\varepsilon$ ?



3. (a) With your partner use the **formal definition** of limit to prove the following:

$$\lim_{x \rightarrow 10} 6x - 7 = 53.$$

Use the same type of careful argument we made in class today with absolute values,  $\varepsilon$ , and  $\delta$ . (See page 8 of the Day 11 online notes for handwritten example.)

- (b) Suppose I told you that  $\varepsilon' = 0.06$ . Use your work above to tell me what  $\delta$  I should use.
- (c) Suppose I told you that  $\varepsilon' = 0.0003$ . Use your work above to tell me what  $\delta$  I should use.