Introduction to Derivatives

5-Minute Review: Instantaneous Rates and Tangent Slope

Recall the analogy that we developed earlier. First we saw that the secant slope of the line through the two points $(a, f(a))$ and $(x, f(x))$ on a curve f was given by the difference quotient.

$$\boxed{\text{difference quotient} = m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}.} \tag{12.1}$$

This is illustrated in the left part of Figure 12.1.

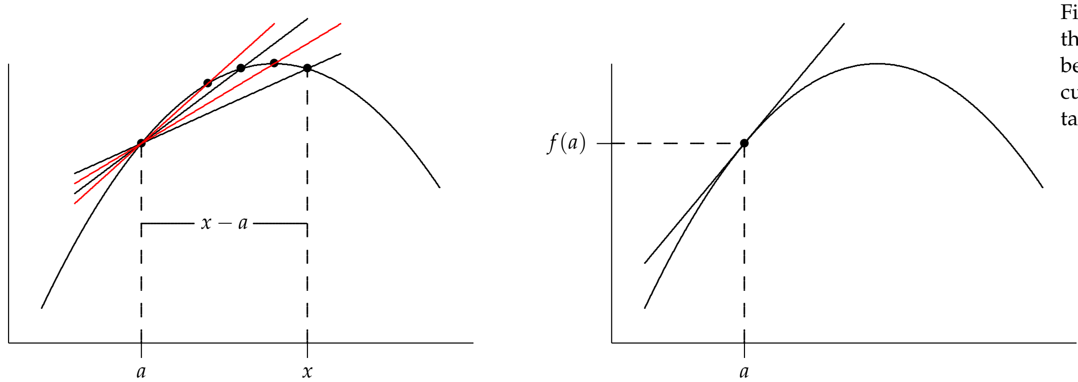


Figure 12.1: Right: The secant lines through $(a, f(a))$ and $(x, f(x))$ as $x \to a$ better approximate the slope of the curve right at $(a, f(a))$. Right: The tangent line that we seek.

Interpretation: If f represents position (distance) and x represents time, then

$$\boxed{\text{difference quotient} = m_{\text{sec}} = \frac{f(x) - f(a)}{x - a} = \frac{\Delta\,\text{position}}{\Delta\,\text{time}} = \text{Average Velocity.}} \tag{12.2}$$

Remember if f represented some other variable quantity, then the difference quotient would simply be the *average rate of change* in that quantity.

Of course we were interested in the the slope of f right at the point $(a, f(a))$ or the instantaneous rate of change right at the time $x = a$. We needed to let x approach a in the formula for the difference quotient. As the interval got shorter, the secant slope approached the tangent slope or the average velocity approached the instantaneous velocity. We could not simply let $x = a$ in the formula for the difference quotient in equation (12.1) because that would produce the meaningless expression $\frac{0}{0}$. It was this indeterminate expression that motivated our discussion of limits. We found that

$$\boxed{m_{\tan} = \lim_{x \to a} m_{\text{sec}} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.} \tag{12.3}$$

Or using the velocity interpretation

$$\boxed{\text{Inst Rate of Change} = \lim_{x\to a} \text{Ave Rate of Change} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}.} \tag{12.4}$$

In short, m_{tan} is the same as the instantaneous rate of change.

DEFINITION 12.1.1 (The Tangent Line). If f is a function defined at point a, then the **tangent slope** of f at $(a, f(a))$ is

$$\boxed{m_{\text{tan}} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a},}$$

if the limit exists. The equation of the **tangent line** is given by

$$\boxed{y - f(a) = m_{\text{tan}}(x-a)}.$$

EXAMPLE 12.1.2. Let $p(x) = 2x^2 + 5x$. Find the slope of the curve at the point $(1, p(1)) = (1, 7)$ and then find the equation of the tangent line there.

Solution. We could graph the function, but we do not need to to carry out the calculations. We simply use Definition 12.1.1

$$\begin{aligned} m_{\text{tan}} = \lim_{x\to a} \frac{p(x)-p(a)}{x-a} &= \lim_{x\to 1} \frac{2x^2+5x-7}{x-1} \\ &= \lim_{x\to 1} \frac{(2x+7)(x-1)}{x-1} = \lim_{x\to 1} 2x+7 \overset{\text{Poly}}{=} 9. \end{aligned}$$

That was easy. Now to determine the equation of the tangent we again use Definition 12.1.1. The formula is

$$y - p(a) = m_{\text{tan}}(x-a).$$

In our case $a = 1$ and $p(1) = 7$ and we just found that $m_{\text{tan}} = 9$. So

$$y - 7 = 9(x-1) \Rightarrow y = 9x - 2.$$

Again, very straightforward if we know the definitions of the terms.

An Alternative Formula for Tangent Slope

In many of the questions that we will shortly face, it is simpler to carry out the calculation of the tangent slope using slightly different notation. It can often greatly simplify the algebra of the difference quotient. We now let $(a, f(a))$ and $(a+h, f(a+h))$ be the two points on the graph of f. Then the difference of the x-coordinates is $(a+h) - a = h$ and the difference in the y-coordinates is $f(a+h) - f(a)$. (See Figure 12.2 and compare it to Figure 12.1.) So the difference quotient becomes

$$\text{difference quotient} = m_{\text{sec}} = \frac{f(a+h)-f(a)}{h}.$$

The denominator is simpler than $x - a$. The penalty is that one must be careful in computing $f(a+h)$. Consequently, we can rewrite Definition 12.1.1 as

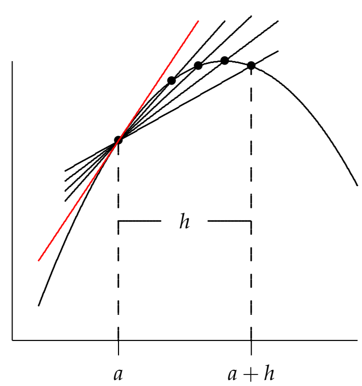


Figure 12.2: Black: The secant lines through $(a, f(a))$ and $(h, f(a+h))$ as $h \to 0$. Red: The tangent line at $(a, f(a))$.

DEFINITION 12.1.3 (The Tangent Line, Alternative Form). If f is a function defined at point a, then the **tangent slope** or **instantaneous rate of change** of f at $(a, f(a))$ is

$$\boxed{m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},}$$

if the limit exists.

EXAMPLE 12.1.4 (Equation of a Tangent). Use the alternative form of the definition of tangent slope to find the equation of the tangent line to $y = f(x) = \sqrt{x}$ at $a = 4$. Then find $m_{\tan}$ at $a = 8$.

Solution. First for the the point at $x = 4$ we need to find $m_{\tan}$. Using Definition 12.1.3, we find

$$\begin{aligned} m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h} \cdot \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2} \\ &= \lim_{h \to 0} \frac{4+h-4}{h(\sqrt{4+h}+2)} \\ &= \lim_{h \to 0} \frac{h}{h(\sqrt{4+h}+2)} \\ &= \lim_{h \to 0} \frac{1}{\sqrt{4+h}+2} \\ &= \frac{1}{4}. \end{aligned}$$

By definition the equation of the tangent line is

$$y - f(a) = m_{\tan}(x - a).$$

In this case $a = 4$, $f(a) = 2$, and $m_{\tan} = \frac{1}{4}$, so

$$y - 2 = \frac{1}{4}(x - 4) \Rightarrow y = \frac{x}{4} + 1.$$

Take a second to look back at the calculation of $m_{\tan}$. The algebra was simplified especially in the denominator and that fact that h is approaching 0.

Now for the the point at $x = 8$ we need to find $m_{\tan}$. Using Definition 12.1.3 again, we find

$$\begin{aligned} m_{\tan} = \lim_{h \to 0} \frac{\sqrt{8+h} - \sqrt{8}}{h} &= \lim_{h \to 0} \frac{\sqrt{8+h} - \sqrt{8}}{h} \cdot \frac{\sqrt{8+h} + \sqrt{8}}{\sqrt{8+h} + \sqrt{8}} \\ &= \lim_{h \to 0} \frac{8+h-8}{h(\sqrt{8+h}+\sqrt{8})} \\ &= \lim_{h \to 0} \frac{h}{h(\sqrt{8+h}+\sqrt{8})} \\ &= \lim_{h \to 0} \frac{1}{\sqrt{8+h}+\sqrt{8}} \\ &= \frac{1}{2\sqrt{8}} = \frac{1}{4\sqrt{2}}. \end{aligned}$$

Notice how similar the calculation was to the one for the point $x = 4$.

YOU TRY IT 12.1. Determine the tangent slope for $y = f(x) = \sqrt{x}$ at $a = 9$. Notice how similar the calculation is to that in Example 12.1.4.

Answer to YOU TRY IT 12.1 : $m_{\tan} = \frac{1}{6}$.

We could continue to find tangent slopes for $f(x) = \sqrt{x}$ at several other points. The calculation is always 'the same' in format, but the numbers are a bit different each time. Is there a way to discern a pattern here so that we can do several of these calculations more quickly? The answers is 'yes' and it will lead us to define a new function associated with the original function f.

The Derivative Function

We saw in Example 12.1.4 that the slope of the curve $f(x) = \sqrt{x}$ changed at the as the point changed. Each time we change the point we had to redo a similar slope calculation. The important idea is that for each point on the curve we could compute the slope of the curve at that point. That means that we have a new function. At each number a in the domain of the original function f, we can ask what is the slope of the tangent line right at $(a, f(a))$? The input to the function is a point a in the domain of f. The output is the corresponding tangent slope at $(a, f(a))$. This new function is called the **derivative** of f because it is derived from the original function.

The derivative function has a special notation. We let f' (read it as 'f *prime*') denote the derivative of f. In Example 12.1.4 where $f(x) = \sqrt{x}$, we found the $f'(4) = \frac{1}{4}$ because $m_{\tan} = \frac{1}{4}$ at the point $(4, f(4))$. Similarly, $f'(8) = \frac{1}{4\sqrt{2}}$. In YOU TRY IT 12.1 $f'(9) = \frac{1}{6}$. Back in Example 12.1.2 where $p(x) = 2x^2 + 5x$, we found that $p'(1) = 9$, that is, the tangent slope at $(1, p(1))$ was 9.

More generally, the derivative function $f'(x)$, when it exists, represents the slope $m_{\tan}$ of the tangent line (or the instantaneous rate of change) at a variable point $(x, f(x))$. Often we can determine a formula for $f'(x)$ by replacing a by the variable x in the calculation for $m_{\tan}$. We call $f'(x)$ the **derivative function**. Specifically

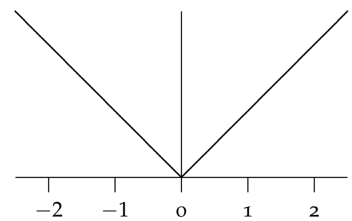


Figure 12.3: If $f(x) = |x|$, then $f'(x)$ is the piecewise function

$$f(x) = \begin{cases} -1, & \text{if } x < 0 \\ 1, & \text{if } x > 1 \end{cases}.$$

because the derivative is the slope of the function.

DEFINITION 12.1.5 (The Derivative)**.** The **derivative** of f is the function defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided that the limit exists. When $f'(x)$ does exist, we say that f is **differentiable** at x. If f is differentiable at every point of an open interval I, we say that f is **differentiable on** I.

EXAMPLE 12.1.6 (A Derivative Function)**.** Let $f(x) = 4x - x^2$. Find $f'(x)$, if it exists.

Solution. Using Definition 12.1.5, the derivative is found by calculating

$$\begin{aligned} f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{\overbrace{4(x+h) - (x+h)^2}^{f(x+h)} - \overbrace{(4x - x^2)}^{f(x)}}{h} \\ &\overset{\text{Expand}}{=} \lim_{h \to 0} \frac{4x + 4h - x^2 - 2xh - h^2 - 4x + x^2}{h} \\ &\overset{\text{Simplify}}{=} \lim_{h \to 0} \frac{4h - 2xh - h^2}{h} \\ &\overset{\text{Cancel}}{=} \lim_{h \to 0} 4 - 2x - h \\ &\overset{\text{Poly}}{=} 4 - 2x \end{aligned}$$

The derivative is $f'(x) = 4 - 2x$ and it exists for all x. This function gives us the slope at every point on the parabola.

Now we can get the slope of the original function $f(x) = 4x - x^2$ at any point we choose without doing a further limit calculation. For example, the slope at 1 is $f'(1) = 4 - 2 = 2$, the slope at 2 is $f'(2) = 4 - 4 = 0$, and the slope at 4 is $f'(4) = 4 - 8 = -4$. These points and the corresponding tangent lines are indicated in Figure 12.5.

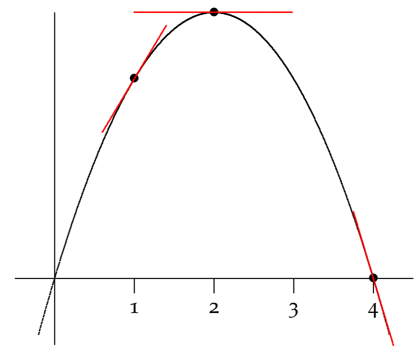


Figure 12.4: Tangent lines to the curve $f(x) = 4x - x^2$. The slope of these tangents is given by the derivative $f'(x) = 2 - 4x$. Notice that some slopes are positive, others negative, and at the top of the curve the slope is 0.

EXAMPLE 12.1.7 (Instantaneous Velocity). Let $s(t) = 4t - t^2$ represent the position of an object (in meters) at time t (in seconds). Find the instantaneous velocity of the object at time $t = 1$, $t = 2$, and $t = 3.5$.

Solution. The instantaneous velocity is the same as the derivative. We just found this derivative in the previous problem (with different notation): $s'(t) = 4 - 2t$. So the instantaneous velocity at $t = 1$ is $s'(1) = 2$ m/s, and similarly $s'(2) = 0$ m/s, and $s'(3.5) = -3$ m/s. The sign of the velocity tells us about the direction of motion: positive is forward or up while negative is backward or down. A velocity of 0 indicates that the object is at rest for the moment (at the top of its flight or changing direction, say).

YOU TRY IT 12.2. Return to YOU TRY IT 12.1 where $f(x) = \sqrt{x}$. Determine $f'(x)$. What is its domain? Use $f'(x)$ to find the tangent slope at $x = 64$. What is the equation of the tangent line at the this same point?

Answers to YOU TRY IT 12.2 : $f'(x) = \frac{1}{2\sqrt{x}}$. $f'(64) = \frac{1}{16}$. $y = \frac{x}{16} + 4$.

EXAMPLE 12.1.8 (Another Derivative Function). Let $f(x) = \dfrac{1}{x^2}$, $(x \neq 0)$. Find $f'(x)$, if it exists.

Solution. Using Definition 12.1.5, the derivative is found by calculating

$$\begin{aligned}
f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{\overbrace{\dfrac{1}{(x+h)^2}}^{f(x+h)} - \overbrace{\dfrac{1}{x^2}}^{f(x)}}{h} \\
&\overset{\text{Com Den}}{=} \lim_{h\to 0} \frac{\frac{x^2-(x+h)^2}{x^2(x+h)^2}}{h} \\
&\overset{\text{Expand}}{=} \lim_{h\to 0} \frac{x^2 - x^2 - 2xh - h^2}{hx^2(x+h)^2} \\
&\overset{\text{Simplify}}{=} \lim_{h\to 0} \frac{-2xh - h^2}{hx^2(x+h)^2} \\
&\overset{\text{Cancel}}{=} \lim_{h\to 0} \frac{-2x - h}{x^2(x+h)^2} \\
&\overset{\text{Rat'l}}{=} -\frac{2x}{x^2(x^2)} \\
&\overset{\text{Simplify}}{=} -\frac{2}{x^3}
\end{aligned}$$

So $f'(x) = -\dfrac{2}{x^3}$ for $x \neq 0$.

YOU TRY IT 12.3. Let $f(x) = \frac{1}{x}$. Determine $f'(x)$. What is its domain? Use $f'(x)$ to find the tangent slope at $x = 2$. What is the equation of the tangent line at the this same point?

Answers to YOU TRY IT 12.2 : $f'(x) = -\frac{1}{x^2}$. $f'(2) = -\frac{1}{4}$. $y = -\frac{x}{4} + \frac{3}{2}$.

Derivatives from Graphs

Often in the 'real world' we have graphs of functions (from data that's been collected) but we do not have a formula for the function—sometimes no simple formula exists. Nonetheless from the graph of the original function we can often obtain a graph of the derivative by interpreting the derivative as the slope of the the original function. Let's start with a relatively easy example that will also introduce the idea points where a function is not differentiable.

EXAMPLE 12.1.9 (Graphical Differentiation)**.** Let $f(x)$ be the piecewise function shown in **black** Figure 12.5.

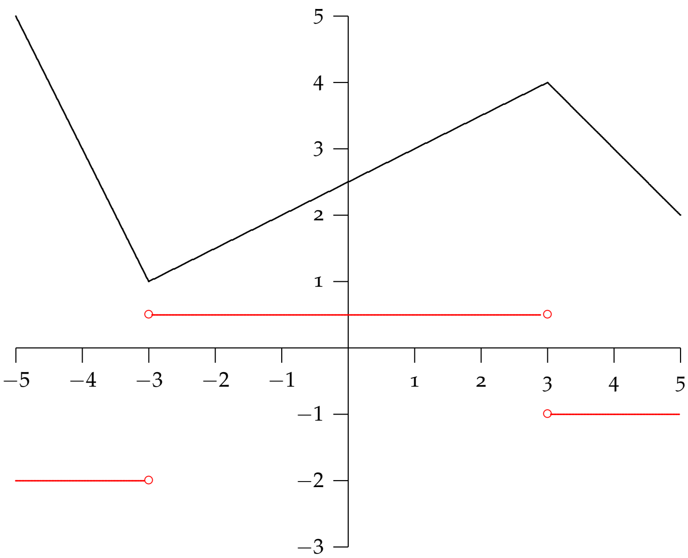


Figure 12.5: Black: The graph of the piecewise function $f(x)$. Red: The derivative $f'(x)$. Note that there are three points where the derivative does not exist. These points correspond to corners in the graph of f where there is not unique tangent slope.

Solution. Remember that the geometric meaning of the derivative is as the tangent slope. Using Figure 12.5, the slope of the section of the graph for $x < -3$ is -2, which means that $f'(x) = -2$ for $x < -2$. Similarly the slope is $f'(x) = \frac{1}{2}$ for $-3 < x < 3$ and $f'(x) = -1$ for $x > 3$.

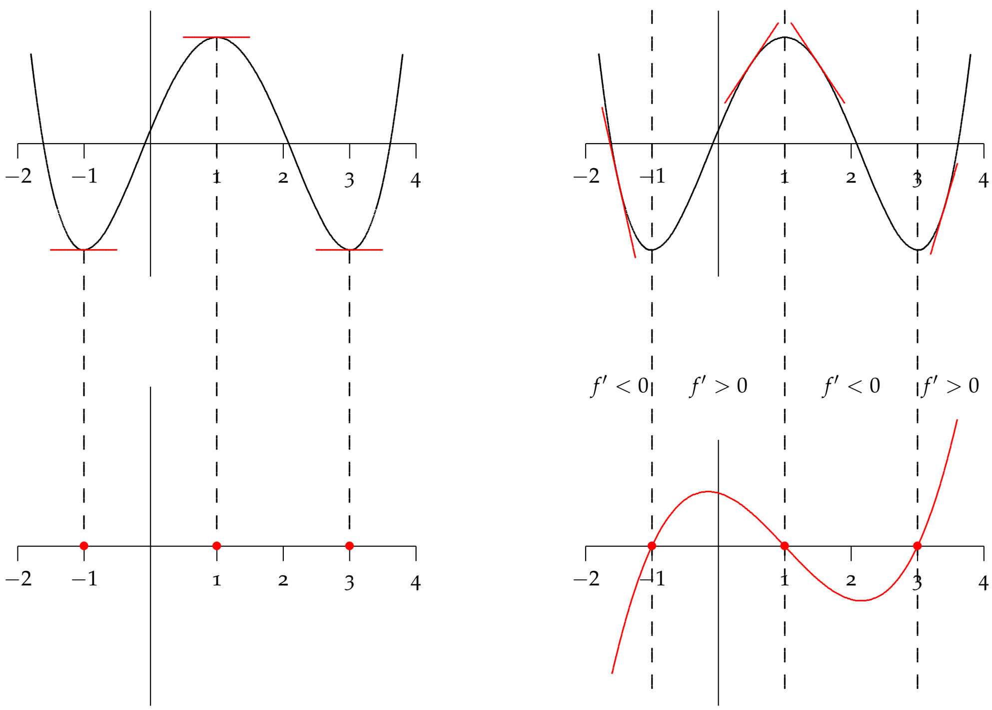
Note: The slope of the tangent line changes abruptly at both $x = -3$ and $x = 3$. There are 'corners' in the graph at these two values. As a result, there is no single value of the slope that makes sense at each point. In other words, $f'(-3)$ and $f'(3)$ do not exist. The derivative is not continuous at these points.

EXAMPLE 12.1.10 (Graphical Differentiation)**.** Sketch the graph of $f'(x)$ if $f(x)$ is the function shown in the upper half of Figure 12.6.

Solution. The graph of f in **black** is repeated in the upper half of Figure 12.6. Without an equation for f, we cannot calculate f' exactly. Remember that the geometric meaning of the derivative is as the tangent slope. The best we can do is estimate the slope of f at various points and use these estimates to plot f'. We proceed in steps to construct the graph of f' in red in the lower half of the figure. The steps are listed to the left.

$f'(x) = 0$ at -1, 0, and 1.

The tangent slopes indicate the sign of f'.



…so $f'(x) = 0$ at -1, 0, and 1.

A rough graph of f'.

Figure 12.6: The process to construct f'.

1. On the left side, we see that f has horizontal tangents at $x = -1$, 1, 3. Horizontal tangents mean the tangent slope (the derivative) is 0. So $f'(-1) = 0$, $f'(1) = 0$, and $f'(3) = 0$. These three points are marked with • on the lower left graph for f'.

2. Now use the graph of f on the upper right.
   (a) For $x < -1$: The tangent slopes to the graph f are negative. So $f' < 0$ when $x < -1$. These slopes increases f' reaches 0 right at $x = -1$.
   (b) For $-1 < x < 1$: The slopes of the tangents are positive. The slope of f starts off small near -1 and increases and then decreases until it gets back to 0 at $x = 1$. We must make the graph of f' do the same.
   (c) For $1 < x < 3$: The slope of a tangent (and so f') is negative. f' decreases at first near -1 and then increases until it gets back to 0 at $x = 3$.
   (d) For $x > 3$: The tangent slope and hence f' are positive and increasing.