Trig Derivatives

In this section we will show that the standard trig functions are differentiable at each point in their respective domains by using the definition of the derivative, some trig identities, and some limit results that we have already worked out.

Two Key Facts

Recall from our earlier work with trig limits that we have previously shown

THEOREM 11.0.1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$

From high school trigonometry, you should recall that

THEOREM 11.0.2. Let x and h be any real numbers. Then

(*a*) $\sin(x+h) = \sin x \cos h + \cos x \sin h$;

(*b*) $\cos(x+h) = \cos x \cos h - \sin x \sin h$.

Some Intuition. Before we actually determine the derivatives of $\sin x$ and $\cos x$, let's see if we can anticipate the results. Let's first look at the graph of $\sin x$.

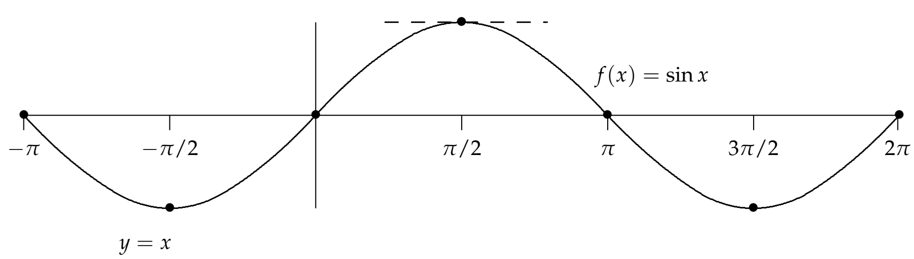

Figure 11.1: The graph of $f(x) = \sin x$.

You already know the derivative of $\sin x$ at $x = 0$. How? Well, remember the definition of the derivative at a point: In this case, because $\sin 0 = 0$,

$$\frac{d}{dx}(\sin x)\Big|_{x=0} = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\sin x - \sin 0}{x - 0} = \lim_{x \to 0} \frac{\sin x}{x} \overset{\text{Thm 11.0.1}}{=} 1.$$

So $\frac{d}{dx}(\sin x)\big|_{x=0} = 1$. Because the graph of $\sin x$ repeats every 2π units (we say the graph is *periodic*), this means that $\frac{d}{dx}(\sin x)\big|_{x=2\pi} = 1$ also.

Next, look at the tangent to $\sin x$ at $x = \pi/2$ in Figure 11.1 (top). The tangent is horizontal, so its slope is 0. This means that the derivative of $\sin x$ at $x = \pi/2$ is 0, that is, $\frac{d}{dx}(\sin x)\big|_{x=\pi/2} = 0$. The same is true at $x = -\pi/2$ and $x = 3\pi/2$. In other words, $\frac{d}{dx}(\sin x)\big|_{x=-\pi/2} = 0$ and $\frac{d}{dx}(\sin x)\big|_{x=3\pi/2} = 0$.

Finally, put a mirror along the vertical line $x = \pi/2$ in Figure 11.1 (top). The points at $x = 0$ and $x = \pi$ are reflected into each other. The graph of $\sin x$ at $x = \pi$ is just the "reversed image" of the graph at $x = 0$. Consequently the slope at $x = \pi$ is -1 instead of 1. So $\frac{d}{dx}(\sin x)\big|_{x=\pi} = -1$. Again, because $\sin x$ repeats every 2π units, $\frac{d}{dx}(\sin x)\big|_{x=-\pi} = -1$. These calculations are summarized below.

$x =$	$-\pi$	$-\pi/2$	0	$\pi/2$	π	$3\pi/2$	2π
$f'(x) = \frac{d}{dx}(\sin x)$	-1	0	1	0	-1	0	1

In Figure 11.2 we have plotted the values of the derivative of $\sin x$ from the table.

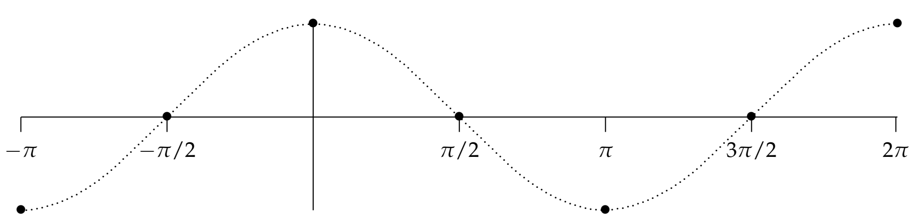

Figure 11.2: The derivative of $\sin x$ at selected points that where it can be easily calculated. What familiar graph passes through all of these points?

What familiar function do you know that goes through all of these points? Why, the cosine function, of course! So it appears that $\frac{d}{dx}(\sin x) = \cos x$. Amazing stuff! Let's see if we can prove it.

THEOREM 11.0.3. The function $f(x) = \sin x$ is differentiable everywhere and

$$\frac{d}{dx}(\sin x) = \cos x.$$

Proof. We use the definition of the derivative along with Theorem 11.0.1 and 11.0.2.

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&\overset{\text{Theorem 11.0.2(a)}}{=} \lim_{h\to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h - \sin x}{h} + \lim_{h\to 0} \frac{\cos x \sin h}{h} \\
&\overset{\text{Simplify}}{=} \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h} \\
&\overset{\text{Theorem 11.0.1}}{=} \sin x \cdot 0 + \cos x \cdot 1 \\
&= \cos x.
\end{aligned}$$

□

YOU TRY IT 11.1. Use exactly the same ideas to prove the following:

THEOREM 11.0.4. The function $f(x) = \cos x$ is differentiable everywhere and

$$\frac{d}{dx}(\cos x) = -\sin x.$$

ANSWER TO YOU TRY IT 11.1. This time

$$\frac{d}{dx}(\cos x) = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \overset{\text{Theorem 11.0.2(a)}}{=} \lim_{h\to 0}\frac{(\cos x\cos h - \sin x \sin h) - \cos x}{h}$$
$$= \lim_{h\to 0}\frac{\cos x\cos h - \cos x}{h} - \lim_{h\to 0}\frac{\sin x \sin h}{h}$$
$$\overset{\text{Simplify}}{=} \cos x \cdot \lim_{h\to 0}\frac{\cos h - 1}{h} - \sin x \cdot \lim_{h\to 0}\frac{\sin h}{h}$$
$$\overset{\text{Theorem 11.0.1}}{=} \cos x \cdot 0 - \sin x \cdot 1 = -\sin x.$$

Using the derivatives of $\sin x$ and $\cos x$ and the quotient rule, we can now find the derivative of $\tan x$ at each point in its domain.

$$\begin{aligned}\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) &\overset{\text{Quotient Rule}}{=} \frac{(\cos x\cos x - \sin x(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &\overset{\text{Trig Id}}{=} \frac{1}{\cos^2 x} \\ &= \sec^2 x.\end{aligned}$$

Thus, we have proven

THEOREM 11.0.5. The function $f(x) = \tan x$ is differentiable everywhere in its domain and

$$\frac{d}{dx}(\tan x) = \sec^2 x.$$

Using the derivatives of $\cos x$ and the rule for derivatives of reciprocal (or quotient) functions, we can find the derivative of $\sec x$ at each point in its domain.

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{0\cdot\cos x - 1\cdot(-\sin x)}{\cos^2 x} = \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} = \sec x \tan x.$$

Thus, we have proven

THEOREM 11.0.6. The function $f(x) = \sec x$ is differentiable everywhere in its domain and

$$\frac{d}{dx}(\sec x) = \sec x \tan x.$$

YOU TRY IT 11.2 (Extra Credit). Use the ideas above to prove the following:

THEOREM 11.0.7. The functions $\cot x$ and $\csc x$ are differentiable everywhere in their respective domains and

$$\frac{d}{dx}(\cot x) = -\csc^2 x \quad \text{and} \quad \frac{d}{dx}(\csc x) = -\csc x \cot x.$$

Examples

We can combine the derivative rules that we just developed for the trig functions with the earlier derivative rules that we have studied. Here are several examples.

EXAMPLE 11.0.8. Determine the derivative of $F(t) = \sin t \cos t$.

SOLUTION. Use the product rule with the derivative rules for $\sin x$ and $\cos x$

$$\frac{d}{dt}(\overbrace{\sin t}^{f(t)}\overbrace{\cos t}^{g(t)}) = \overbrace{\cos t}^{f'(t)}\overbrace{\cos t}^{g(t)}) + \overbrace{\sin t}^{f(t)}\overbrace{(-\sin t)}^{g'(t)} = \cos^2 t - \sin^2 t.$$

EXAMPLE 11.0.9. Determine the derivative of $F(x) = \frac{e^x+1}{\tan x}$.

SOLUTION. Use the quotient rule with the derivative rules for e^x and $\tan x$

$$\frac{d}{dx}\left(\frac{\overbrace{e^x+1}^{f(x)}}{\underbrace{\tan x}_{g(x)}}\right) = \frac{\overbrace{e^x}^{f'(x)}\overbrace{\tan x}^{g(x)}) - \overbrace{(e^x+1)}^{f(x)}\overbrace{\sec^2 x}^{g'(x)}}{\underbrace{\tan^2 x}_{(g(x))^2}}.$$

EXAMPLE 11.0.10. Determine the derivative of $\sin^2 t$.

SOLUTION. We don't yet have a method (yet, but soon!) for taking the derivative of the square of a function. However, we can think of this as product.

$$\frac{d}{dt}(\sin^2 t) = \frac{d}{dt}(\overbrace{\sin t}^{f(t)}\overbrace{\sin t}^{g(t)}) = \overbrace{\cos t}^{f'(t)}\overbrace{\sin t}^{g(t)} + \overbrace{\sin t}^{f(t)}\overbrace{(\cos t)}^{g'(t)} = 2\sin t\cos t.$$

EXAMPLE 11.0.11. Determine the derivative of $F(x) = \frac{x^2\sin x}{e^x}$.

SOLUTION. The function $f(x)$ is a quotient where the numerator is a product. So when taking the derivative of the numerator we will need to use the product rule.

$$\frac{d}{dx}\left(\frac{\overbrace{x^2\sin x}^{f(x)}}{\underbrace{e^x}_{g(x)}}\right) = \frac{\overbrace{2x\sin x + x^2\cos x}^{f'(x)}\overbrace{e^x}^{g(x)}) - \overbrace{((x^2\sin x))}^{f(x)}\overbrace{e^x}^{g'(x)}}{\underbrace{(e^x)^2}_{(g(x))^2}} = \frac{x(2\sin x + x\cos x - x\sin x)}{e^x}.$$

YOU TRY IT 11.3. Use the derivative rules that we have developed to determine the indicated derivatives. Be sure to simplify your answers.

(a) $D_x\left(\frac{3x}{e^{4x}+2x}\right)$ (b) $\frac{d}{dx}(9\sin x + e^{2x})$ (c) $\frac{d}{dx}(9\sin x\cos x)$

(d) $\frac{d}{dx}\left(\frac{e^{-3x}+1}{\sin x}\right)$ (e) $\frac{d}{dx}\left(\frac{\sin x}{1+\cos x}\right)$ (f) $\frac{d}{dx}\left(\frac{x+1}{x^2e^x}\right)$

ANSWER TO YOU TRY IT 11.3. Simplified answers:

(a) $\frac{3e^{4x}(1-4x)}{(e^{4x}+2x)^2}$ (b) $9\cos x + 2e^{2x}$ (c) $9(\cos^2 x - \sin^2 x)$

(d) $\frac{-e^{-3x}(3\sin x + \cos x) - \cos x}{\sin^2 x}$ (e) $\frac{1}{1+\cos x}$ (f) $-\frac{x^2+2x+2}{x^3e^x}$

YOU TRY IT 11.4. Use the derivative rules that we have developed to determine the indicated derivatives. Be sure to simplify your answers.

(a) $y = x + \frac{1-\cos x}{2+\sin x}$ (b) $f(x) = \frac{1}{4x^2} + \frac{5}{\sqrt{x^5}} + \frac{9}{5x}$ (c) $g(x) = x^2\sec(x)$

(d) $f(x) = \frac{e^{-3x}\tan x}{6}$ (e) $y = -23\pi + \sqrt[7]{t^2}\cos t$ (f) $f(t) = \frac{t\tan t}{t^2+2}$

(g) $y = \frac{6\sin x}{x^{3/2}}$ (h) $f(\theta) = \frac{3\theta^2-1}{3\cos(\theta)}$ (i) $\sec^2 z$

ANSWER TO YOU TRY IT 11.4.

(a) $y' \stackrel{\text{Quot}}{=} 1 + \dfrac{\sin x(2+\sin x) - (1-\cos x)(-\cos x)}{(2+\sin x)^2} = 1 + \dfrac{2\sin x - \cos x + \sin^2 x - \cos^2 x}{(2+\sin x)^2}$

(b) $d_x\left(\frac{1}{4}x^{-2} + 5x^{5/2} + \frac{9}{5}x^{-1}\right) = -\frac{1}{2}x^{-3} - \frac{25}{2}x^{-7/2} - \frac{9}{5}x^{-2}$.

(c) $g'(x) \stackrel{\text{Prod}}{=} 2x\sec(x) + x^2\sec(x)\tan(x) = x\sec x(2 + x\tan x)$.

(d) Product rule, not quotient. $D_x\left(\frac{1}{6}e^{-3x}\tan x\right) \stackrel{\text{Prod}}{=} \frac{1}{6}[-3e^{-3x}\tan x + e^{-3x}\sec^2 x] = \frac{e^{-3x}}{6}(-3\tan x + \sec^2 x)$.

(e) $D_t(-23\pi + t^{2/7}\cos t) \stackrel{\text{Prod}}{=} \frac{2}{7}t^{-5/7}\cos t - t^{2/7}\sin t$.

(f) $y' = \dfrac{(\tan t + t\sec^2 t)(t^2+2) - (t\tan t)2t}{(t^2+2)^2} = \dfrac{(t^3+2t)\sec^2 t + (2-t^2)\tan t}{(t^2+2)^2}$.

(g) $y' \stackrel{\text{Quot}}{=} \dfrac{6x^{3/2}\cos x - 9x^{1/2}\sin x}{x^3} = \dfrac{6x\cos x - 9\sin x}{x^{5/2}}$ [Divide out $x^{1/2}$.]

(h) $f'(\theta) \stackrel{\text{Quot}}{=} \dfrac{6\theta(3\cos\theta) + (3\theta^2-1)3\sin\theta}{(3\cos\theta)^2} = \dfrac{18\theta\cos\theta + 3(3\theta^2-1)\sin\theta}{(3\cos\theta)^2}$.

(i) $y' = D_z(\sec^2 z) = D_z[(\sec z)(\sec z)] \stackrel{\text{Prod}}{=} \sec z\tan z(\sec z) + \sec z(\sec z\tan z) = 2\sec^2 z\tan z$.

YOU TRY IT 11.5. Let $f(x) = \sin(2x)$. Obviously $2x = x + x$. Use the addition formula for the sine function to convert $f(x) = \sin(2x)$ into a product and then find $f'(x)$.

ANSWER TO YOU TRY IT 11.5. $D_x[\sin(2x)] = D_x[\sin(x+x)] = D_x[\sin x\cos x + \cos x\sin x] = D_x[2\sin x\cos x] = 2\cos x\cos x - 2\sin x\sin x = 2\sin^2 x - 2\cos^2 x$.

In the next section, we will state a rule which will allow us to compute the derivative of $\sin(2x)$ without having to use a trig identity. This general rule will allow us to compute the derivative of much more general composite functions.