The Chain Rule

In the previous section we had to use a trig identity to determine the derivative of . $h(x) = \sin(2x)$. We can view $h(x)$ as the *composition* of two functions. Let $g(x) = 2x$ and $f(x) = \sin x$. Then $h(x) = f(g(x))$. We know the derivatives of both f and g, so can we write the derivative of $f(g(x))$ in terms of the f and g and their derivatives? The answer is, "Yes," as we will see below.

Composition Review

Recall that the composition $(f \circ g)(x) = f(g(x))$ has domain all x in the domain of g such that $g(x)$ is in the domain of f. Think of $g(x)$ as the *inside function* and f as the *outside function*.

Here's the general problem. We start with the composite function $y = f(g(x))$. Determine

$$\frac{dy}{dx} = \frac{d}{dx}\left[f(g(x))\right]$$

in terms of f and g. To help us "see" the composition more clearly, let $u = g(x)$ for the inside function. Now we may write

$$y = f(g(x)) = f(u).$$

EXAMPLE 16.0.1. In the compositions below, identify the inside function $u = g(x)$ and the outside function $f(u)$.

(*a*) $y = \sin(x^3)$ (*b*) $y = \sqrt{x^2 + 5x + 1}$ (*c*) $y = (1 + \tan x)^3$ (*d*) $e^{\sin x}$

SOLUTION. Analyzing the compositions

(*a*) $y = \sin(x^3)$: inside $u = g(x) = x^3$ and outside $f(u) = \sin u$.
(*b*) $y = \sqrt{x^2 + 5x + 1}$: inside $u = g(x) = x^2 + 5x + 1$ and outside $f(u) = \sqrt{u}$.
(*c*) $y = (1 + \tan x)^3$: inside $u = g(x) = 1 + \tan x$ and outside $f(u) = u^3$.
(*d*) $y = e^{\sin x}$: inside $u = g(x) = \sin x$ and outside $f(u) = e^u$.

Intuition

Here's the intuition for how the chain rule for derivatives of compositions works. Suppose that (Y)an does her calculus homework twice as fast at (U)rsala. Written in terms of rates of change, this means that $\dfrac{dY}{dU} = 2$.

Next assume that (U)rsala is 1.5 times faster than (X)avier, so $\dfrac{dU}{dX} = 1.5$.

How many times faster is (Y)an than (X)avier? In other words, what is $\dfrac{dY}{dX}$? Well, we multiple the two rates together:

$$\frac{dY}{dX} = 2(1.5) = 3 \text{ times faster than (X)avier.}$$

Writing this symbolically,

$$\frac{dY}{dX} = \frac{dY}{\cancel{dU}} \cdot \frac{\cancel{dU}}{dX} = 2(1.5) = 3.$$

Notice it is as if we can cancel the dU's in the product of the rates of change to calculate $\frac{dY}{dX}$.

Intuition Applied to Composites

In our case we want to calculate

$$\frac{dy}{dx} = \frac{d}{dx}\left[f(g(x))\right] = \frac{d}{dx}\left[f(u)\right].$$

Here's where our composition notation helps. The inside function is $u = g(x)$ so

$$\frac{du}{dx} = g'(x). \tag{16.1}$$

Next, $y = f(u)$. Taking the derivative of y with respect to u (not x), we have

$$\frac{dy}{du} = f'(u) = f'(g(x)). \tag{16.2}$$

The chain rule says that we can multiply these two derivatives to get

$$\frac{dy}{dx} = \frac{dy}{\cancel{du}} \cdot \frac{\cancel{du}}{dx}.$$

Though we have not proved this result (take Math 331), we will use this result. Stating it carefully,

THEOREM 16.0.2 (Chain Rule)**.** Suppose that $u = g(x)$ is differentiable at x and $y = f(x)$ is differentiable at $u = g(x)$. Then the composite $f(g(x))$ is differentiable at x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Using (16.1) and (16.2) this can be written as

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))g'(x).$$

Examples

We can combine the chain rule with the earlier derivative rules that we have studied. Here are several examples.

EXAMPLE 16.0.3. Determine $\frac{d}{dx}\left[\cos(3x^2+1)\right]$.

SOLUTION. Rewrite the composition $u = g(x) = 3x^2+1$ and $y = f(u) = \cos u$. So $\frac{du}{dx} = g'(x) = 6x$ and $\frac{dy}{du} = f'(u) = -\sin u$. Using the chain rule (version 1)

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \overbrace{(-\sin u)}^{\frac{dy}{du}}\overbrace{(6x)}^{\frac{du}{dx}} = -\sin(3x^2+1)(6x) = -6x\sin(3x^2+1).$$

Remember to substitute for u back in terms of x and to simplify the form of the expression at the last steps.

If we use version 2 of the chain rule instead,

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))g'(x) = \overbrace{-\sin(g(x))}^{f'(g(x))}\overbrace{(6x)}^{g'(x)} = -\sin(3x^2+1)(6x) = -6x\sin(3x^2+1).$$

EXAMPLE 16.0.4. Determine $\frac{d}{dx}\left[\tan(e^{-4x})\right]$.

SOLUTION. Rewrite the composition $u = g(x) = e^{-4x}$ and $y = f(u) = \tan u$. So $\frac{du}{dx} = g'(x) = -4e^{-x}$ and $\frac{dy}{du} = f'(u) = \sec^2 u$. Using the chain rule (version 1)

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \overbrace{(\sec^2 u)}^{\frac{dy}{du}}\overbrace{(e^{-4x})}^{\frac{du}{dx}} = (\sec^2 e^{-4x})(-4e^{-4x}) = -4e^{-4x}\sec^2 e^{-4x}.$$

If we use version 2 of the chain rule instead,

$$\frac{d}{dx}\left[f(g(x))\right] = \overbrace{\sec^2(g(x))}^{f'(g(x))}\overbrace{(-4e^{-4x})}^{g'(x)} = \sec^2(e^{-4x})(-4e^{-4x}) = -4e^{-4x}\sec^2 e^{-4x}.$$

EXAMPLE 16.0.5. Determine $\frac{d}{dx}\left[(x^2+x)^8\right]$.

SOLUTION. Rewrite the composition $u = g(x) = x^2 + x$ and $y = f(u) = u^8$. So $\frac{du}{dx} = g'(x) = 2x+1$ and $\frac{dy}{du} = f'(u) = 8u^7$. Using the chain rule (version 1)

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \overbrace{(8u^7)}^{\frac{dy}{du}}\overbrace{(2x+1)}^{\frac{du}{dx}} = 8(x^2+x)^7(2x+1) = (16x+8)(x^2+x)^7.$$

Using version 2 of the chain rule instead,

$$\frac{d}{dx}\left[f(g(x))\right] = \overbrace{8(g(x))^7}^{f'(g(x))}\overbrace{(2x+1)}^{g'(x)} = 8(x^2+x)^7(2x+1) = (16x+8)(x^2+x)^7.$$

EXAMPLE 16.0.6. Determine $\frac{d}{dx}\left[\sqrt{6x^2+10)}\right]$.

SOLUTION. Rewrite the composition $u = g(x) = 6x^2 + 10$ and $y = f(u) = \sqrt{u}$. So $\frac{du}{dx} = g'(x) = 12x$ and $\frac{dy}{du} = f'(u) = \frac{1}{2}u^{-1/2}$. Using version 1 of the chain rule

$$\frac{dy}{dx} = \overbrace{\left(\frac{1}{2}u^{-1/2}\right)}^{\frac{dy}{du}}\overbrace{(12x)}^{\frac{du}{dx}} = \frac{1}{2}(6x^2+10)^{-1/2}(12x) = 6x(6x^2+10)^{-1/2}.$$

Using version 2 of the chain rule instead,

$$\frac{d}{dx}\left[f(g(x))\right] = \overbrace{\frac{1}{2}(g(x))}^{f'(g(x))}\overbrace{(12x)}^{g'(x)} = \frac{1}{2}(6x^2+10)^{-1/2}(12x) = 6x(6x^2+10)^{-1/2}.$$

EXAMPLE 16.0.7. Here's a combined chain and quotient rule problem. Determine $\frac{d}{dx}\left[\left(\frac{x+1}{x^2+7}\right)^7\right]$.

SOLUTION. Rewrite the composition $u = g(x) = \frac{x+1}{x^2+7}$ and $y = f(u) = u^7$. Using version 1 of the chain rule

$$\frac{dy}{dx} = \overbrace{\left(7u^6\right)}^{\frac{dy}{du}}\overbrace{\left(\frac{1(x^2+7)-(x+1)x}{(x^2+7)^2}\right)}^{\frac{du}{dx}} = 7\left(\frac{x+1}{x^2+7}\right)^6\left(\frac{-x^2-2x+7}{(x^2+7)^2}\right) = \frac{-7(x^2+2x-7)(x+1)^6}{(x^2+7)^8}.$$

YOU TRY IT 16.1. Fill in the following table to determine the derivatives of the given composite functions.

Composite: $y = f(g(x))$	Outside: $y = f(u)$	Outside Deriv: $\frac{dy}{du} = f'(u)$	Inside: $u = g(x)$	Inside Deriv: $\frac{du}{dx} = g'(x)$	Chain Rule: $\frac{dy}{du} \cdot \frac{du}{dx}$ $f'(u)\frac{du}{dx} = f'(g(x))g'(x)$
$\sin(3x^2+1)$					
$\cos(e^{2x})$					
$\tan(10x)$					
$(3x^2+1)^5$					
$\sin^5 x = (\sin x)^5$					
$\sqrt{3x^2+1}$					
$\frac{1}{3x^2+1}$					
$\left(\frac{x+5}{3x+1}\right)^4$					
e^{3x^2+1}					

ANSWER TO YOU TRY IT 16.1. Simplified final answers:

(a) $6x\cos(3x^2+1)$ (b) $-2e^{2x}\sin(e^{2x})$ (c) $10\sec^2(10x)$
(d) $30x(3x^2+1)^4 3x(3x^2+1)^{-1/2}$ (e) $5\cos x\sin^4 x$ (f) $3x(3x^2+1)^{-1/2}$
(g) $-6x(3x^2+1)^{-2}$ (h) $-\frac{56(x+5)^3}{(3x+1)^5}$ (i) $6xe^{3x^2+1}$

EXAMPLE 16.0.8. Sometimes we must apply the chain rule more than once in a problem. The idea is to work from the outside and keep taking derivatives of the inner functions as we encounter them Determine the derivative of $\tan(e^{7x^{-}x})$.

SOLUTION. Apply the chain rule a step at a time.

$$\frac{d}{dx}\left[\tan\overbrace{(e^{7x^2-x})}^{u}\right] = \overbrace{7\sec^2(e^{7x^2-x})}^{f'(u)} \cdot \overbrace{\frac{d}{dx}(e^{7x^2-x})}^{\frac{du}{dx}} = 7\sec^2(e^{7x^{-}x})(e^{7x^2-x})(14x^2-1).$$

EXAMPLE 16.0.9. Sometimes we must apply the chain rule more than once in a problem. The idea is to work from the outside and keep taking derivatives of the inner functions as we encounter them. Determine the derivative of $\tan(e^{7x^{-}x})$.

SOLUTION. Apply the chain rule a step at a time.

$$\frac{d}{dx}\left[\tan\overbrace{(e^{7x^2-x})}^{u}\right] = \overbrace{7\sec^2(e^{7x^2-x})}^{f'(u)} \cdot \overbrace{\frac{d}{dx}(e^{7x^2-x})}^{\frac{du}{dx}} = 7\sec^2(e^{7x^{-}x})(e^{7x^2-x})(14x^2-1).$$

EXAMPLE 16.0.10. Here's another triple play chain rule problem. Determine the derivative of $f(g(h(x))) = e^{\sqrt{2x^4+x^2+1}}$.

SOLUTION. Apply the chain rule a step at a time. The outer function is $f(u) = e^u$, the middle function is $g(w) = \sqrt{w}$ and the inner function is $h(x) = 2x^4 + x^2 + 1$. Just keep taking derivatives from the outside in:

$$\frac{d}{dx}[e^{\sqrt{2x^4+x^2+1}}] = \overbrace{\left(e^{\sqrt{2x^4+x^2+1}}\right)}^{f'(u)}\overbrace{\left[\frac{1}{2}(2x^4+x^2+1)^{-1/2}\right]}^{g'(w)}\overbrace{(8x^3+2x)}^{h'(x)} = \frac{(4x^3+x)e^{\sqrt{2x^4+x^2+1}}}{\sqrt{2x^4+x^2+1}}.$$

YOU TRY IT 16.2. Use the derivative rules that we have developed to determine the indicated derivatives. Be sure to simplify your answers.

(*a*) $\sec(3x^2) + 1$	(*b*) $\tan(\pi x + 1)$	(*c*) e^{2s^4+s}
(*d*) $\dfrac{\sin(e^{5x})}{4}$	(*e*) $\tan^4 x$	(*f*) $\tan(x^4)$
(*g*) $(e^{2x}\cos x)^3$	(*h*) $3e^{t^2 \sin t}$	(*i*) $\csc^2 z$

ANSWER TO YOU TRY IT 16.2. Simplified final answers:

(*a*) $6x \sec(3x^2) tan(3x^2)$	(*b*) $\pi \sec^2(\pi x + 1)$	(*c*) $(8s^3 + 1)e^{2s^4+s}$
(*d*) $\dfrac{5e^{5x}\cos(e^{5x})}{4}$	(*e*) $4\tan^3 x \sec^2 x$	(*f*) $4x^3 \sec^2(x^4)$
(*g*) $3(e^{2x}\cos x)^2 e^{2x}(2\cos x - \sin x)$	(*h*) $3e^{t^2 \sin t}(2t \sin t + t^2 \cos t)$	(*i*) $-\csc^2 z \cot z$