Implicit Differentiation

Implicit Differentiation—Using the Chain Rule

In the previous section we focused on the derivatives of composites and saw that

THEOREM 20.1.1 (Chain Rule)**.** Suppose that $u = g(x)$ is differentiable at x and $y = f(x)$ is differentiable at $u = g(x)$. Then the composite $f(g(x))$ is differentiable at x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

This can be written as

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))g'(x).$$

Notice that we can apply the chain rule even if we do not know the precise *formula* for the inner function.

EXAMPLE 20.1.2. Write out the chain rule derivative for each these, assuming u and v are functions of x.

(*a*) $\frac{d}{dx}(6\sin(u)) = 6\cos u\frac{du}{dx}$. Note the since u is some unknown function of x, we must include the chain rule derivative ('inside derivative'), du/dx.

(*b*) $\frac{d}{dx}[e^u] = e^u\frac{du}{dx}$ (*c*) $\frac{d}{dx}[\sin u] = \cos u\frac{du}{dx}$ (*d*) $\frac{d}{dx}[\tan u] = \sec^2 u\frac{du}{dx}$

(*e*) $\frac{d}{dx}[\sec u] = \sec u \tan u\frac{du}{dx}$ (*f*) $\frac{d}{dx}[u^n] = nu^{n-1}\frac{du}{dx}$

(*g*) $\frac{d}{dx}\left(u^3\cos(v)\right) = 3u^2 \cdot \frac{du}{dx} \cdot \cos(v) + u^3(-\sin(v))\frac{dv}{dx} = 3u^2\cos(v)\frac{du}{dx} - u^3\sin(v)\frac{dv}{dx}$.

Notice that the symbol (letter) we use for the inner function does not matter. For example, if we use y for the inside function instead of u, we have $y = g(x)$, then

$$\frac{d}{dx}[\overbrace{(g(x))^3}^{y^3}] = 3y^2\frac{dy}{dx} = 3(g(x))^2g'(x)\ldots \text{ or } \frac{d}{dx}(y^3) = 3y^2\frac{dy}{dx}.$$

Similarly

$$\frac{d}{dx}[\overbrace{(g(x))^5}^{y^5}] = 5u^4\frac{dy}{dx} = 5(g(x))^4g'(x)\ldots \text{ or } \frac{d}{dx}(y^5) = 5y^4\frac{dy}{dx}.$$

More generally, you text states

THEOREM 20.1.3 (General Power Rule)**.** If $f(x)$ is a differentiable function, then

$$\frac{d}{dx}[(f(x))^n] = n(f(x))^{n-1}f'(x).$$

Now what we need to pay attention to is the variables used in the function and the variable that the derivative is taken with respect to. Notice that

$$\frac{d}{dx}(u^9) = 9u^8\frac{du}{dx}.$$

Similarly

$$\frac{d}{dx}(y^9) = 9y^8\frac{dy}{dx} \qquad \text{and} \qquad \frac{d}{dt}(x^9) = 9x^8\frac{dx}{dt}.$$

However

$$\frac{d}{dx}(x^9) = 9x^8,$$

because the variables are the same and there is no chain rule 'inside derivative.'

EXAMPLE 20.1.4. Here are several other examples:

(1) $\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx}$ (2) $\frac{d}{dx}(\tan y) = \sec^2 y \frac{dy}{dx}$

(3) $\frac{d}{dx}(\tan x) = \sec^2 x$ (4) $\frac{d}{dt}(a^2) = 2a\frac{da}{dt}$

5. If $c^2 = a^2 + b^2$, then $\frac{d}{dt}(c^2) = \frac{d}{dt}(a^2 + b^2)$ so $2c\frac{dc}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt}$.

6. If that $A = \pi r^2$, then $\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$ so $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$. Here we are assuming that the radius of the circle is changing with respect to time. In other words, we assume that r is some (unknown) function of t.

EXAMPLE 20.1.5. Fancier examples: This time assume u, v, and y are functions of x.

1. Using the product rule, $\frac{d}{dx}(u^2v^3) = 2u\frac{du}{dx}v^3 + u^2 3v^2\frac{dv}{dx}$.

2. And $\frac{d}{dx}(x^2y^3) = 2xy^3 + x^2 3y^2\frac{dy}{dx}$.

Implicit and Explicit Functions

Sometimes we have an explicit formula for a function. For example, $y = f(x) = e^x$ or $y = f(x) = x^2 + 3x + 1$ defines y explicitly as a function of x. For each value of x that we put into either function, we obtain a single output value.

Compare this to the equation $xy^2 = 1$. Here the relation between x and y is defined implicitly. Rather than a function we have an equation involving both x and y. In this case we could solve for y and obtain

$$y^2 = \frac{1}{x} \text{ which implies } y = \frac{1}{\sqrt{x}} \text{ or } y = -\frac{1}{\sqrt{x}}.$$

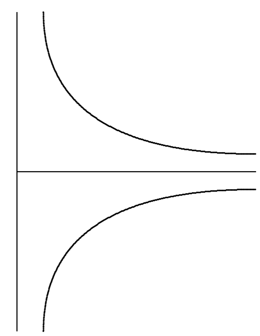


Figure 20.1: The graph of the relation $xy^2 = 1$ is not the graph of a function. It does not pass the vertical line test.

The result is not a function.

We will take as our definition that an **implicit function** is an *equation involving two or more variables*.

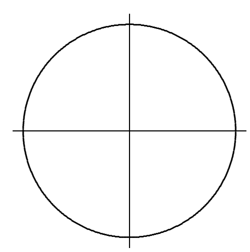


Figure 20.2: The graph of the relation $x^2 + y^2 = 1$ is not the graph of a function. It does not pass the vertical line test.

EXAMPLE 20.1.6. Consider the implicit function given by $x^2 + y^2 = 1$. We know that its graph is a circle and is not an ordinary function since its graph does not pass the vertical line test.

In this case, we could create two ordinary (explicit) functions from this single implicit function. Solving for y we find that $y^2 = 1 - x^2$ so that $y = \sqrt{1 - x^2}$ or $y = -\sqrt{1 - x^2}$.

EXAMPLE 20.1.7. Here's another implicitly defined function: $x^3 + yx^2 + y^3 = 7$. This time there is no simple way to make one of more explicit functions from this relation.

The Goal

Given an implicit function in variables x and y, find the derivative $\frac{dy}{dx}$ without ever solving for y explicitly. This is called **implicit differentiation**. The next example illustrates how to do this.

EXAMPLE 20.1.8. Determine $\dfrac{dy}{dx}$ if $x^2 + y^2 = 1$. Then calculate the slope of the curve at the points $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, $(0, 1)$, an $(1, 0)$.

SOLUTION. Note first that since we want to determine $\frac{dy}{dx}$, we are treating x as the independent variable. So we are taking the derivative with respect to x. Thus,

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1)$$

so

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0.$$

Using the chain rule

$$2x + 2y\frac{dy}{dx} = 0.$$

Now solve for $\frac{dy}{dx}$

$$2y\frac{dy}{dx} = -2x$$

so

$$\frac{dy}{dx} = -\frac{x}{y}, \ (y \neq 0).$$

Now we can determine slopes at specific points on the circle. The slope at $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ —check that this point is on the circle— is given by substituting in the particular x and y values of the point into the expression for $\frac{dy}{dx}$. Notice the notation used to indicate the point at which the derivative is calculated.

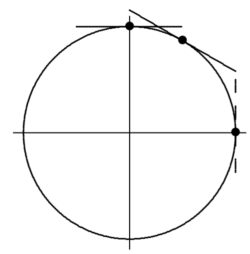


Figure 20.3: The tangents at $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(0, 1)$.

$$\left.\frac{dy}{dx}\right|_{(\frac{1}{2}, \frac{\sqrt{3}}{2})} = -\left.\frac{x}{y}\right|_{(\frac{1}{2}, \frac{\sqrt{3}}{2})} = -\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}.$$

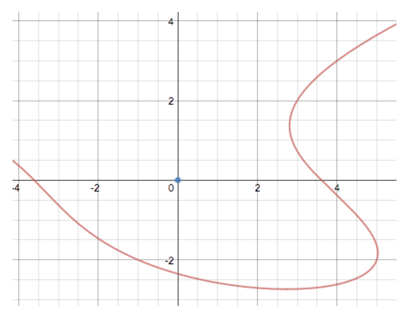
At $(0, 1)$

$$\left.\frac{dy}{dx}\right|_{(0,1)} = -\left.\frac{x}{y}\right|_{(0,1)} = -\frac{0}{1} = 0.$$

Looking at the graph of the circle, these two slopes look to be correct. Finally, the slope at $(1, 0)$ is not defined since $\dfrac{dy}{dx} = -\dfrac{x}{y}$, $(y \neq 0)$. The tangent line is vertical.

EXAMPLE 20.1.9 (Like an exam question). Let $x^2 + 2xy = y^3 + 13$.

1. Find $\dfrac{dy}{dx}$
2. Verify that $(3, 2)$ is on the curve.
3. Determine the equation of the tangent line at $(3, 2)$.



The graph of $x^2 + 2xy = y^3 + 13$.

SOLUTION. (1) Take the derivative (with respect to x) of both sides of the equation.

$$\frac{d}{dx}(x^2 + 2xy) = \frac{d}{dx}(y^3 + 13)$$

Using the product and chain rules

$$2x + 2y + 2x\frac{dy}{dx} = 0 + 3y^2\frac{dy}{dx}$$

Gather all the $\frac{dy}{dx}$ terms on the left side

$$2x\frac{dy}{dx} - 3y^2\frac{dy}{dx} = -2x - 2y$$

$$(2x - 3y^2)\frac{dy}{dx} = -2x - 2y$$

Solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{-2x - 2y}{2x - 3y^2}$$

2. Check that $(3,2)$ is on the curve:

$$x^2+2xy = 3^2+2(3)(2) = 21 \quad \text{and} \quad 13+y^3 = 13+2^3 = 21.$$

Thus $(3,2)$ is on the curve.

3. Determine the tangent at $(3,2)$. The slope is

$$m = \left.\frac{dy}{dx}\right|_{(3,2)} = \left.\frac{-2x-2y}{2x-3y^2}\right|_{(3,2)} = \frac{-2(3)-2(2)}{2(3)-3(2^2)} = \frac{-10}{-6} = \frac{5}{3}.$$

So the equation is

$$y-2 = \tfrac{5}{3}(x-3) \quad \text{or} \quad y = \tfrac{5}{3}x-5+2 \quad \text{or} \quad y = \tfrac{5}{3}x-3.$$

EXAMPLE 20.1.10 (Like an exam question). Let $x^2+x^2y^2 = y^3-3$.

1. Verify that $(1,2)$ is on the curve.
2. Determine the equation of the tangent line at $(1,2)$.

SOLUTION. 1. Check that $(1,2)$ is on the curve. At $(1,2)$

$$x^2+x^2y^2 = 1^2+1^2 2^2 = 5 \quad \text{and} \quad y^3-3 = 2^3-3 = 5.$$

Thus $(1,2)$ is on the curve.

2. To find the tangent at $(1,2)$ we need the slope; we already have the point. We need to find $\frac{dy}{dx}$. Take the derivative (with respect to x) of both sides of the equation.

$$\frac{d}{dx}(x^2+x^2y^2) = \frac{d}{dx}(y^3+13)$$
$$2x+2xy^2+2x^2y\frac{dy}{dx} = 3y^2\frac{dy}{dx}$$

Gather all the $\frac{dy}{dx}$ terms on the left side

$$(2x^2y-3y^2)\frac{dy}{dx} = -2x-2xy^2$$

Solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{-2x-2xy^2}{2x^2y-3y^2}$$

Evaluate $\frac{dy}{dx}$ at $(1,2)$ to get the slope

$$m = \left.\frac{dy}{dx}\right|_{(1,2)} = \left.\frac{-2x-2xy^2}{2x^2y-3y^2}\right|_{(1,2)} = \frac{-2(1)-2(1)(2^2)}{2(1^2)(2)-3(2^2)} = \frac{-10}{-8} = \frac{5}{4}.$$

So the equation is

$$y-2 = \tfrac{5}{4}(x-1) \quad \text{or} \quad y = \tfrac{5}{4}x-\tfrac{5}{4}+2 \quad \text{or} \quad y = \tfrac{5}{4}x+\tfrac{3}{4}.$$

EXAMPLE 20.1.11 (Easier). Let $x+y^2-4y = 2$.

1. Determine the equations of all the tangents when $x = 2$.
2. At what points are the tangents vertical.

SOLUTION. 1. First we must determine the y-coordinates of the points on the curve that have $x = 2$ as the first coordinate. Since $x+y^2-4y = 2$, when $x = 2$

$$2+y^2-4y = 2 \quad \text{so} \quad y^2-4y = y(y-4) = 0 \quad \text{or} \quad y = 0,\ 4.$$

So the points are $(2,0)$ and $(2,4)$.

2. To find the tangents we need the slope so we need to find $\frac{dy}{dx}$. Take the derivative (with respect to x) of both sides of the equation.

$$\begin{aligned} \frac{d}{dx}(x+y^2-4y) &= \frac{d}{dx}(2) \\ 1+2y\frac{dy}{dx}-4\frac{dy}{dx} &= 0 \\ (2y-4)\frac{dy}{dx} &= -1 \end{aligned}$$

Solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{-1}{2y-4} \qquad (y \neq 2)$$

At $(2,0)$: Evaluate $\dfrac{dy}{dx}$ at $(2,0)$ to get the slope at the first point

$$m = \frac{dy}{dx}\bigg|_{(2,0)} = \frac{-1}{2y-4}\bigg|_{(2,0)} = \frac{-1}{2(0)-4} = \frac{1}{4}.$$

So the equation is

$$y-0 = \tfrac{1}{4}(x-2) \quad \text{or} \quad y = \tfrac{1}{4}x - \tfrac{1}{2}.$$

See Figure 20.4. At $(2,4)$:

$$m = \frac{dy}{dx}\bigg|_{(2,4)} = \frac{-1}{2y-4}\bigg|_{(2,4)} = \frac{-1}{2(4)-4} = -\frac{1}{4}.$$

So the equation is

$$y-4 = -\tfrac{1}{4}(x-2) \quad \text{or} \quad y = -\tfrac{1}{4}x - \tfrac{9}{2}.$$

See Figure 20.4.

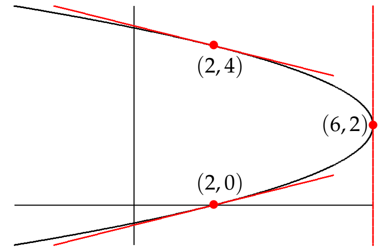


Figure 20.4: The graph of the relation$x+y^2-4y=2$.

3. The graph will have a vertical tangent where the slope $\dfrac{dy}{dx} = \dfrac{-1}{2y-4}$ becomes infinite. This is where the denominator goes to 0. This is at $y=2$. When $y=2$,

$$x+y^2-4y=2 \quad \text{so} \quad x+2^2-4(2)=2 \quad \text{so} \quad x=2-4+8=6.$$

Thus, there is a vertical tangent at $(6,2)$. See Figure 20.4.

EXAMPLE 20.1.12. Find $\frac{dy}{dx}$ if $\cos y + \sin x = x^2$.

SOLUTION. Take the derivative with respect to x and solve for $\frac{dy}{dx}$.

$$\begin{aligned} \frac{d}{dx}(\cos y + \sin x) &= \frac{d}{dx}(x^2) \\ -\sin y\frac{dy}{dx} + \cos x &= 2x \\ -\sin y\frac{dy}{dx} &= 2x - \cos x \\ \frac{dy}{dx} &= -\frac{2x-\cos x}{\sin y} = \frac{\cos x - 2x}{\sin y}. \end{aligned}$$

EXAMPLE 20.1.13. Find $\frac{dy}{dx}$ if $e^{xy} = x$.

SOLUTION. Take the derivative with respect to x of both sides of the equation and solve for $\frac{dy}{dx}$.

$$\begin{aligned} \frac{d}{dx}(e^{xy}) &= \frac{d}{dx}(x) \\ e^{xy}\left(y + x\frac{dy}{dx}\right) &= 1 \\ e^{xy}y + e^{xy}x\frac{dy}{dx} &= 1 \\ e^{xy}x\frac{dy}{dx} &= 1 - e^{xy}y \\ \frac{dy}{dx} &= \frac{1-e^{xy}y}{e^{xy}x} \end{aligned}$$

EXAMPLE 20.1.14. Find $\frac{dy}{dx}$ if $\tan(xy) = x^2 + y$.

SOLUTION. Take the derivative with respect to x of both sides of the equation and solve for $\frac{dy}{dx}$.

$$\begin{aligned}
\frac{d}{dx}(\tan(x^3y^3)) &= \frac{d}{dx}(x^2+y) \\
\sec^2(xy)\left(3x^2y^3 + 3x^3y^2\frac{dy}{dx}\right) &= 2x + \frac{dy}{dx} \\
3x^2y^3\sec^2(xy) + 3x^3y^2\sec^2(xy)\frac{dy}{dx} &= 2x + \frac{dy}{dx} \\
3x^3y^2\sec^2(xy)\frac{dy}{dx} - \frac{dy}{dx} &= 2x - 3x^2y^3\sec^2(xy) \\
(3x^3y^2\sec^2(xy) - 1)\frac{dy}{dx} &= 2x - 3x^2y^3\sec^2(xy) \\
\frac{dy}{dx} &= \frac{2x - 3x^2y^3\sec^2(xy)}{3x^3y^2\sec^2(xy) - 1}
\end{aligned}$$

YOU TRY IT 20.1. Determine $\frac{dy}{dx}$ given $e^{x+2y} = x$.

ANSWER TO YOU TRY IT 20.1. $\frac{dy}{dx} = \frac{1}{2e^{x+2y}} - \frac{1}{2}$.

YOU TRY IT 20.2. Determine $\frac{dy}{dx}$ given $e^y + 9y = 4x^5 + 1$.

ANSWER TO YOU TRY IT 20.2. $\frac{dy}{dx} = \frac{20x^4}{e^y + 9}$.

YOU TRY IT 20.3. Determine the equation of the tangent line to $\frac{dy}{dx}$ given $e^y + 9y = 4x^5 + 1$ at $(1, 2)$.

ANSWER TO YOU TRY IT ??. $\frac{dy}{dx} = \frac{-y^3 - 2xy}{3xy^2 + x^2}$. Tangent: $y - 2 = -\frac{5}{13}(x - 1)$.