Inverse Functions

Review from Last Time: The Derivative of $y = \ln x$

Last time we saw that

THEOREM 22.0.1. The natural log function is differentiable and

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

More generally, the chain rule version is

$$\boxed{\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot \frac{du}{dx}.}$$

EXAMPLE 22.0.2. Determine $D_x\left[(\ln(2x+1))^5\right]$.

SOLUTION. Use the chain rule.

$$D_x[(\ln(2x+1))^5] = D_x(u^5) = 5u^4\frac{du}{dx} = 5(\ln(2x+1))^4 \cdot \frac{1}{2x+1} \cdot 2 = \frac{10[\ln(2x+1)]^4}{2x+1}.$$

An Important Special Case. Since we can only take logs of positive numbers, often times we use the log of an absolute value, e.g., $\ln|x|$. We can find the derivatives of such expressions as follows.

$$D_x[\ln|x|] = \begin{cases} D_x[\ln x] & \text{if } x > 0, \\ D_x[\ln(-x)] & \text{if } x < 0 \end{cases} = \begin{cases} \frac{1}{x} & \text{if } x > 0, \\ \frac{1}{-x}(-1) = \frac{1}{x} & \text{if } x < 0 \end{cases} = \frac{1}{x} \text{ if } x \neq 0$$

In other words, we get the 'same rule' as without the absolute value:

THEOREM 22.0.3. For $x \neq 0$,

$$D_x(\ln|x|) = \frac{1}{x}$$

The chain rule version when u is a function of x is

$$\boxed{\frac{d}{dx}(\ln|u|) = \frac{1}{u} \cdot \frac{du}{dx}.}$$

EXAMPLE 22.0.4. Here's one that involves a number of log properties:

$$\begin{aligned} D_t\left[\ln\left|\frac{e^t\cos t}{\sqrt{t^2+1}}\right|\right] &= D_t\left[\ln e^t + \ln|\cos t| - \ln\sqrt{t^2+1}\right| \\ &= D_t\left[t + \ln|\cos t| - \tfrac{1}{2}\ln|t^2+1|\right] \\ &= 1 + \frac{1}{\cos t}\cdot(-\sin t) - \frac{2t}{2(t^2+1)} \\ &= 1 - \tan t - \frac{t}{t^2+1} \end{aligned}$$

YOU TRY IT 22.1. Try finding these derivatives. Use log rules to simplify the functions before taking the derivative.

(a) $D_x\left[\ln|6x^3\sin x|\right]$

(b) $D_x\left[6x^3\ln|\sin x|\right]$ (different)

(c) $D_x\left[\ln\left|\dfrac{x^4-1}{x^2+1}\right|\right]$

(d) $D_t\left[\ln(t^{(e^t)})\right]$

(e) $D_x\left[\ln\sqrt[3]{3x^3+x+1}\right]$

(f) $D_s\left[\ln\left(5^{\ln s}\right)\right]$

ANSWER TO YOU TRY IT 22.1. Use Theorem 22.0.3 and log properties, then simplify.

(a) $D_x\left[\ln|6x^3\sin x|\right] = D_x\left[\ln|6x^3| + \ln|\sin x|\right] = \dfrac{3}{x} + \cot x$

(b) $18x^3\ln|\sin x| + 6x^3\cot x$

(c) $D_x\left[\ln\left|\dfrac{x^4-1}{x^2+1}\right|\right] = D_x\left[\ln\left|x^4-1\right| - \ln\left|x^2+1\right|\right] = \dfrac{4x^3}{x^4-1} - \dfrac{2x}{x^2+1}$

(d) $D_t\left[\ln(t^{(e^t)})\right] = D_t\left[e^t\ln t\right] = e^t\ln t + \dfrac{e^t}{t}$

(e) $D_x\left[\ln\sqrt[3]{3x^3+x+1}\right] = D_x\left[\dfrac{1}{3}\cdot\ln\sqrt[3]{3x^3+x+1}\right] = \dfrac{1}{3}\dfrac{9x^2+1}{3x^3+x+1}$

(f) $D_s\left[\ln\left(5^{\ln s}\right)\right] = D_s\left[\ln s\ln 5\right] = \dfrac{\ln 5}{s}$

In part (f), remember that $\ln 5$ is just a constant.

The Derivative of Other Exponential Functions: $y = b^x$

When we first determined the derivative of e^x we were looking at general exponential functions of the form $y = f(x) = b^x$. Remember that we picked out e to be the number so that $\lim_{h\to 0} \dfrac{e^h - 1}{h} = 1$ which then made computing the derivative of e^x very easy. But what about all the other exponential functions of the form $y = b^x$? Is there a way to determine their derivatives? Indeed, there is! And it turns out to be pretty easy to do.

Method 1: This first development of the derivative of $y = b^x$ here is different than in your text. We will use implicit differentiation. Suppose that $b > 0$ and $y = f(x) = b^x$. Our goal is to find $\dfrac{dy}{dx} = f'(x) = \dfrac{d}{dx}(b^x)$.

Start with	$y = b^x$
Take the logs of both sides	$\ln y = \ln b^x$
Simplify	$\ln y = x \overbrace{\ln b}^{\text{constant}}$
Take the derivative	$\dfrac{d}{dx}(\ln y) = \dfrac{d}{dx} x \ln b$
	$\dfrac{1}{y}\dfrac{dy}{dx} = \ln b$
Solve	$\dfrac{dy}{dx} = y \ln b$
Substitute back for y	$\dfrac{dy}{dx} = b^x \ln b$

That was easy. So we have

THEOREM 22.0.5 (Derivative of b^x). If $b > 0$, then for all x,

$$\boxed{\frac{d}{dx}(b^x) = b^x \ln b.}$$

The chain rule version when u is a function of x is

$$\boxed{\frac{d}{dx}(b^u) = b^u \ln b \frac{du}{dx}}.$$

Method 2: We could also determine the derivative of $y = b^x$ using inverse functions. Because e^x and $\ln x$ are inverse functions, one undoes the other, so if we apply both of them in succession, we end up with the original input.

$$\ln(e^x) = x \qquad \text{and} \qquad e^{\ln(x)} = x.$$

In particular,

$$b^x = e^{\ln b^x} = e^{x \ln b}$$

where we used a log property at the last step to bring the power out front. So b^x is just an exponential function of the form e^{kx} where $k = \ln b$ is a constant. [1] So

$$\frac{d}{dx}(b^x) = \frac{d}{dx}\left(e^{x \ln b}\right) = \ln b e^{x \ln b} = (\ln b) b^x = \boxed{b^x \ln b}. \tag{22.1}$$

We get the same result as above.

[1] This is the great advantage of working with logs: Logs turn products into sums and powers into products. These simplify many calculations.

EXAMPLE 22.0.6. Find the derivatives of the following functions.

(*a*) $y = 2^x$ (*b*) $z = 15(3^{t/10})$ (*c*) $y = 5^{t^3 \sin t}$ (*d*) $y = 4^x \tan(4x)$

Solution. Using Theorem 22.0.5,

(*a*) $\dfrac{d}{dx}\left(2^x\right) = 2^x \ln 2$.

(*b*) This time we use the chain rule:

$$\frac{d}{dt}\left(15(3^{t/10})\right) = 15(3^{t/10}) \ln 3 \cdot \overbrace{\tfrac{d}{dt}\left(\tfrac{t}{10}\right)}^{\frac{1}{10}} = \tfrac{3}{2}(3^{t/10}) \ln 3.$$

(*c*) Use the chain rule in combination with Theorem 22.0.5,

$$\frac{d}{dt}(5^{\overbrace{(t^3 \sin t)}^{u}}) = 5^{t^3 \sin t} \ln 5 \cdot \overbrace{(3t^2 \sin t - t^3 \cos t)}^{\frac{d}{dt}\left(t^3 \sin t\right)}.$$

(*d*) Use the product rule:

$$\frac{d}{dx}\left(4^x \tan(4x)\right) = 4^x \ln 4 \cdot \tan(4x) + 4^x \sec^2(4x) \cdot 4 = 4^x(\ln 4 \tan(4x) + 4\sec^2(4x)).$$

Logarithmic Differentiation

There are still types of functions that we have not tried to differentiate yet. Sometimes we can make use of our existing techniques and clever algebra to find derivatives of very complicated functions. **Logarithmic differentiation** refers to the process of first taking the natural log of a function $y = f(x)$, then solving for the derivative $\dfrac{dy}{dx}$. On the surface of it, it would seem that logs would only make a complicated function *more* complicated. But remember that logs turn powers into products and products into sums. That's the key.

Here's a neat problem to illustrate the idea.

EXAMPLE 22.0.7. Use the chain rule and implicit differentiation along with logs to find the derivative of $y = f(x) = x^x$.

Solution. We begin by taking the natural log of both sides and simplifying using log properties.

$$\ln y = \ln x^x \overset{\text{Powers}}{=} x \ln x.$$

Remember we want to find $\dfrac{dy}{dx}$, so take the derivative of both sides (implicitly on the left).

$$\begin{aligned} \frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln x) \Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} &= 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln(x) + 1 \\ \frac{dy}{dx} &\overset{\text{Solve}}{=} y[\ln(x) + 1] \\ \frac{dy}{dx} &\overset{\text{Substitute}}{=} x^x[\ln(x) + 1] \end{aligned}$$

In other words, we have shown that $\dfrac{d}{dx}(x^x) = x^x[\ln(x) + 1]$. Neat! Easy!

Here are a couple more.

EXAMPLE 22.0.8. Find the derivative of $y = (1+x^2)^{\tan x}$.

Solution. Take the natural log of both sides and simplify using log properties.

$$\ln y = \ln(1+x^2)^{\tan x} \overset{\text{Powers}}{=} \tan x \ln(1+x^2).$$

Take the derivative of both sides (implicitly on the left) and solve for $\frac{dy}{dx}$.

$$\begin{aligned}
\frac{d}{dx}(\ln y) &= \frac{d}{dx}\left(\tan x \ln(1+x^2)\right) \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \sec^2 x \ln(1+x^2) + \tan x \cdot \frac{2x}{1+x^2} \\
\frac{dy}{dx} &\overset{\text{Solve}}{=} y\left[\sec^2 x \ln(1+x^2) + \frac{2x\tan x}{1+x^2}\right] \\
\frac{dy}{dx} &\overset{\text{Substitute}}{=} \ln(1+x^2)^{\tan x}\left[\sec^2 x \ln(1+x^2) + \frac{2x\tan x}{1+x^2}\right]
\end{aligned}$$

So $\frac{d}{dx}\left(\ln(1+x^2)^{\tan x}\right) = \ln(1+x^2)^{\tan x}\left[\sec^2 x \ln(1+x^2) + \frac{2x\tan x}{1+x^2}\right]$. Not bad!

EXAMPLE 22.0.9. Find the derivative of $y = (\ln x)^{x^3}$.

Solution. Be careful. This function is NOT the same as $\ln(x^{x^3})$ which would equal $x^3 \ln x$. Instead, take the natural log of both sides and simplify using log properties.

$$\ln y = \ln(\ln x)^{x^3} \overset{\text{Powers}}{=} x^3 \ln(\ln x).$$

Do you see the difference when compared to $\ln(x^{x^3})$

Take the derivative of both sides (implicitly on the left) and solve for $\frac{dy}{dx}$.

$$\begin{aligned}
\frac{1}{y}\cdot\frac{dy}{dx} &= 3x^2\ln(\ln x) + x^3\cdot\frac{1}{\ln x}\cdot\frac{1}{x} \\
\frac{1}{y}\cdot\frac{dy}{dx} &\overset{\text{Solve}}{=} y\left[3x^2\ln(\ln x) + \frac{x^3}{x\ln x}\right] \\
\frac{dy}{dx} &\overset{\text{Substitute}}{=} (\ln x)^{x^3}\left[3x^2\ln(\ln x) + \frac{x^3}{x\ln x}\right]
\end{aligned}$$

Logs can also be used to simplify products and quotients.

EXAMPLE 22.0.10. Find the derivative of $y = \frac{(x^2-1)^5\sqrt{1+x^2}}{x^4+4}$.

Solution. Use logarithmic differentiation to avoid a complicated quotient rule derivative Take the natural log of both sides and then simplify using log properties.

$$\begin{aligned}
\ln y &= \ln\left(\frac{(x^2-1)^5\sqrt{1+x^2}}{x^4+4}\right) \\
&\overset{\text{Log Prop}}{=} \ln(x^2-1)^5 + \ln(1+x^2)^{1/2} - \ln(x^4+4) \\
&\overset{\text{Log Prop}}{=} 5\ln(x^2-1) + \frac{1}{2}\ln(1+x^2) - \ln(x^4+4).
\end{aligned}$$

Take the derivative of both sides and solve for $\frac{dy}{dx}$.

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{10x}{x^2-1}+\frac{x}{1+x^2}-\frac{4x^3}{x^4+4}$$
$$\frac{dy}{dx} \overset{\text{Solve}}{=} y\left[\frac{10x}{x^2-1}+\frac{x}{1+x^2}-\frac{4x^3}{x^4+4}\right]$$
$$\frac{dy}{dx} \overset{\text{Substitute}}{=} \frac{(x^2-1)^5\sqrt{1+x^2}}{x^4+4}\left[\frac{10x}{x^2-1}+\frac{x}{1+x^2}-\frac{4x^3}{x^4+4}\right]$$

That would have been a real mess to do with the quotient rule (which would also require the product rule and the chain rule).

Problems

The following questions will be on the lab tomorrow or are future WebWorK problems. Get a head start.

1. Find the derivatives of the following functions. Use logarithmic differentiation where helpful.

(*a*) $y = (\sin x)^x$ (*b*) $y = x^{\sin x}$ (*c*) $(\sin x)^{\sin x}$ (*d*) $(\arcsin x)^{x^2}$ (*e*) $\left(1+\frac{1}{x}\right)^x$

2. Find the derivatives of these functions using the derivative formula for a general exponential function that we developed before Exam II. (See Theorem 3.18 on page 194).

(*a*) $5\cdot 6^x$ (*b*) $2^x\cot x$ (*c*) $x^\pi+\pi^x$ (*d*) $x^4\cdot 4^x$

(*e*) For which values of x does $x^4\cdot 4^x$ have a horizontal tangent?

Answers.

1. (*a*) $\ln y = \ln(\sin x)^x = x\ln(\sin x) \Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \ln(\sin x)+\frac{x\cos x}{\sin x} \Rightarrow \frac{dy}{dx} = (\sin x)^x(\ln(\sin x)+x\cot x)$.

(*b*) $\ln y = \ln x^{\sin x} = \sin x\ln x \Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \cos x\ln x+(\sin x)\frac{1}{x} \Rightarrow \frac{dy}{dx} = x^{\sin x}\left(\cos x\ln x+\frac{\sin x}{x}\right)$.

(*c*) $\ln y = \ln(\sin x)^{\sin x} = \sin x\ln(\sin x) \Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \cos x\ln(\sin x)+(\sin x)\frac{\cos x}{\sin x} \Rightarrow \frac{dy}{dx} = (\sin x)^{\sin x}\cos x[\ln(\sin x)+1]$.

(*d*) $\ln y = \ln(\arcsin x)^{x^2} = x^2\ln(\arcsin x) \Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = 2x\ln(\arcsin x)+x^2\frac{1}{\arcsin x}\frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{dy}{dx} = (\arcsin x)^{x^2}\left(2x\ln(\arcsin x)+\frac{x^2}{(\arcsin x)\sqrt{1-x^2}}\right)$.

(*e*) $\ln y = \ln\left(1+\frac{1}{x}\right)^x = x\ln\left(1+\frac{1}{x}\right) \Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \ln\left(1+\frac{1}{x}\right)+x\cdot\frac{1}{\left(1+\frac{1}{x}\right)}\cdot\frac{-1}{x^2} \Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \ln\left(1+\frac{1}{x}\right)-\frac{1}{x\left(1+\frac{1}{x}\right)} \Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \ln\left(1+\frac{1}{x}\right)-\frac{1}{(x+1)} \Rightarrow \frac{dy}{dx} = \left(1+\frac{1}{x}\right)^x\left[\ln\left(1+\frac{1}{x}\right)-\frac{1}{(x+1)}\right]$.

2. (*a*) $\frac{d}{dx}[5\cdot 6^x] = 5\cdot 6^x\ln 6 = 5\ln 6(6^x)$.

(*b*) $\frac{d}{dx}[2^x\cot x] = 2^x\ln 2\cot x-2^x\csc^2 x = 2^x[\ln 2\cot x-\csc^2 x]$.

(*c*) $\frac{d}{dx}[x^\pi+\pi^x] = \pi x^{\pi-1}+\pi^x\ln\pi$ (*d*) $\frac{d}{dx}[x^4\cdot 4^x] = 4x^3\cdot 4^x+x^4\cdot 4^x\ln 4 = x^3\cdot 4^x[4+x\ln 4]$

(*e*) From the previous part, the slope is 0 when $x^3\cdot 4^x[4+x\ln 4] = 0$. Therefore $x = 0$ or $x = -\frac{4}{\ln 4}$.