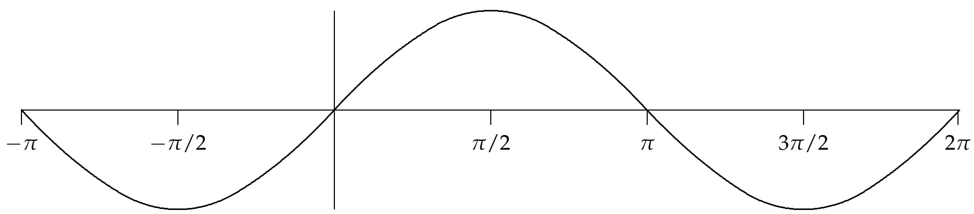
Inverse Trig Functions

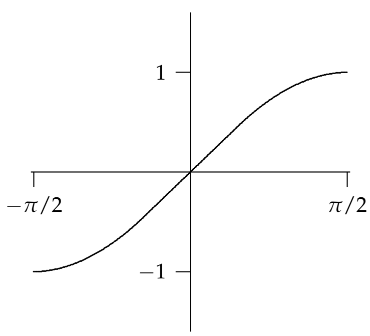
Introduction to Inverse Trig Functions

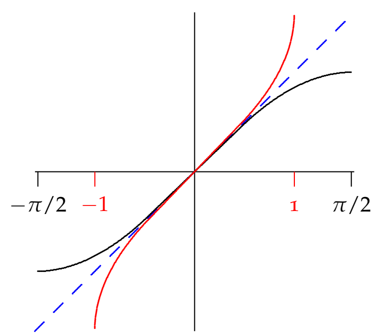
None of the trig functions have inverses because none of them pass the horizontal line test. Their values repeat every 2π units or every π units (tangent, cotangent).



The Inverse Sine Function

However, if we restrict the domain of the sine function (or any of the other trig functions) we can make the function one-to-one on the restricted interval. The figure on the left below shows $\sin x$ restricted to the interval $[-\pi/2, \pi/2]$ where it is, indeed, one-to-one (passes HLT). So it has an inverse there, which we have graphed in red the figure on the right.





The inverse sine function is denoted by $\arcsin x$. Your text uses $\sin^{-1} x$, but most students find $\arcsin x$ less confusing, and that's what we will generally use in this course. Since the domain and range of the sine and inverse sine functions are interchanged, we have

- the domain of $\arcsin x$ is the range of the restricted $\sin x$: $[-1, 1]$.
- the range of $\arcsin x$ is the domain of the restricted $\sin x$: $[-\pi/2, \pi/2]$. This is very important. It says that the output of the inverse sine function is a number (an angle) between $-\pi/2$ and $\pi/2$.

Notice since the arcsine function undoes the sine function, we get some familiar values: $\arcsin(-1) = -\pi/2$ since $\sin(-\pi/2) = -1$. Or $\arcsin(1/2) = \pi/6$ since $\sin(\pi/6) = 1/2$. Or $\arcsin(\sqrt{3}/2) = \pi/3$ since $\sin(\pi/3) = \sqrt{3}/2$.

EXAMPLE 24.1.1. Normally when we calculate $f^{-1}(f(x))$ we get x because the two functions undo each other. The same is true here, if the domain of $\sin x$ is appropriately restricted to $[-\pi/2, \pi/2]$. For example,

$$\arcsin(\sin(\pi/4)) = \arcsin(\sqrt{2}/2) = \pi/4.$$

But if we take a value outside of the restricted domain $[-\pi/2, \pi/2]$ of the sine function

$$\arcsin(\sin(3\pi/4)) = \arcsin(\sqrt{2}/2) = \pi/4.$$

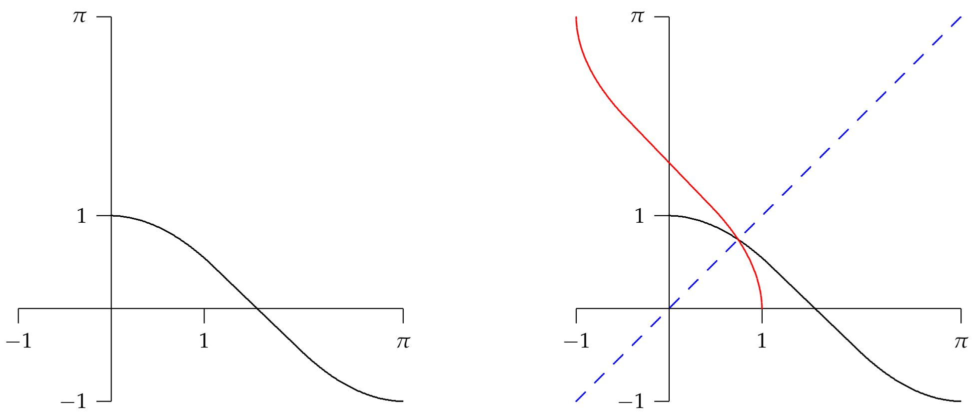
Or

$$\arcsin(\sin(3\pi)) = \arcsin(0) = 0.$$

The two functions do not undo each other since the arcsine function can only return values (or angles) between $-\pi/2$ and $\pi/2$.

The Inverse Cosine Function

We can restrict the domains of the other trig functions so that they, too, have inverses. The figure on the left below shows $\cos x$ restricted to the interval $[0, \pi]$ where it is, indeed, one-to-one. So it has an inverse there, which we have graphed in red the figure on the right.



The inverse cosine function is denoted by $\arccos x$. Since the domain and range of the cosine and inverse cosine functions are interchanged, we have

- the domain of $\arccos x$ is the range of the restricted $\cos x$: $[-1, 1]$.
- the range of $\arccos x$ is the domain of the restricted $\cos x$: $[0, \pi]$.

EXAMPLE 24.1.2. Again we have to be careful about calculating the composites of these inverse functions. They are only inverses when the inputs are in the correct domains. For example,

$$\arccos(\cos(\pi/4)) = \arccos(\sqrt{2}/2) = \pi/4.$$

But if we take a value outside of the restricted domain $[0, \pi]$ of the cosine function

$$\arccos(\cos(-\pi/4)) = \arccos(\sqrt{2}/2) = \pi/4.$$

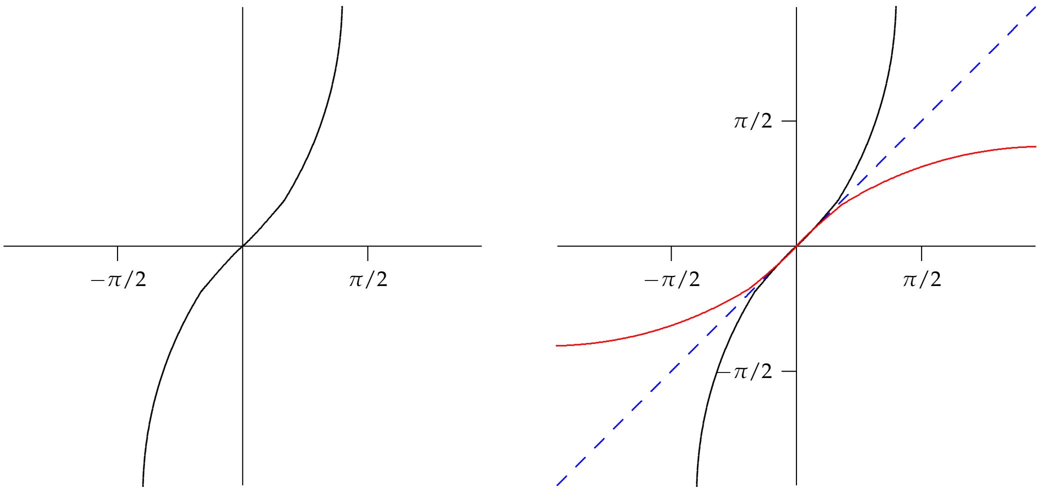
Or

$$\arccos(\cos(3\pi)) = \arccos(-1) = \pi.$$

The two functions do not always undo each other since the inverse cosine function can only return values between 0 and π.

The Inverse Tangent Function

The figure on the left below shows $\tan x$ restricted to the interval $(-\pi/2, \pi/2)$ where it is, indeed, one-to-one. So it has an inverse there, which we have graphed in red the figure on the right.



The inverse tangent function is denoted by $\arctan x$. Since the domain and range of the tangent and inverse tangent functions are interchanged, we have

- the domain of $\arctan x$ is the range of the restricted $\tan x$: $(-\infty, \infty)$.
- the range of $\arctan x$ is the domain of the restricted $\tan x$: $(-\pi/2, \pi/2)$.

EXAMPLE 24.1.3. Again we have to be careful about calculating the composites of these inverse functions. They are only inverses when the inputs are in the correct domains. For example,

$$\arctan(\tan(\pi/4)) = \arctan(1) = \pi/4.$$

But if we take a value outside of the restricted domain $(-\pi/2, \pi/2)$ of the tangent function

$$\arctan(\tan(3\pi/4)) = \arctan(-1) = -\pi/4.$$

Or

$$\arctan(\tan(3\pi)) = \arctan(0) = 0.$$

The two functions do not always undo each other since the inverse tangent function can only return values between $-\pi/2$ and $\pi/2$.

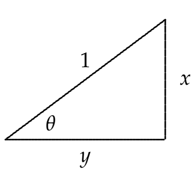
We will concentrate only on the the three inverse functions discussed above. I will leave it to you to read about the other inverse trig functions in your text.

Evaluation Using Triangles

Drawing appropriate right triangles can help evaluate complicated expressions involving the inverse trig functions.

EXAMPLE 24.1.4. Evaluate $\cos(\arcsin x)$.

Solution. Remember that $\arcsin x = \theta$ where θ is just the angle whose sine is x. We want the cosine of this same angle. So let's draw a right triangle with angle θ whose sine is x. Since the sine function is $\frac{\text{opp}}{\text{hyp}}$ we can use the triangle below.



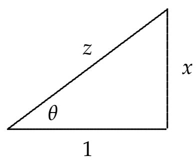
$$x^2 + y^2 = 1 \Rightarrow y = \sqrt{1 - x^2}.$$

Notice $\sin\theta = \frac{x}{1} = x$. So $\arcsin x = \theta$. (θ is the angle whose sine is x.) So

$$\cos(\arcsin x) = \cos(\theta) = \frac{y}{1} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}.$$

EXAMPLE 24.1.5. Evaluate $\sec(\arctan x)$.

Solution. This time we draw a triangle whose tangent is x.



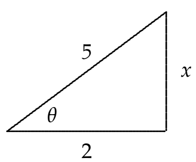
$$z^2 = 1^2 + x^2 \Rightarrow z = \sqrt{1+x^2}.$$

So

$$\sec(\arctan x) = \sec(\theta) = \frac{z}{1} = \sqrt{1+x^2}.$$

EXAMPLE 24.1.6. Evaluate $\sin(\arccos 2/5)$.

Solution. This time we draw a triangle whose cosine is $2/5$.



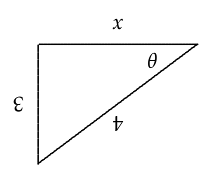
$$2^2 + x^2 = 5^2 \Rightarrow x = \sqrt{5^2 - 2^2} = \sqrt{21}.$$

So

$$\sin(\arccos 2/5) = \sin(\theta) = \frac{x}{5} = \frac{\sqrt{21}}{5}.$$

YOU TRY IT 24.1. Evaluate $\cos(\arcsin 3/4)$.

ANSWER TO YOU TRY IT 24.1. Draw a triangle whose sine is $3/4$.



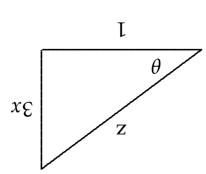
$$3^2 + x^2 = 4^2 \Rightarrow x = \sqrt{4^2 - 3^2} = \sqrt{7}.$$

So

$$\cos(\arcsin 3/4) = \cos(\theta) = \frac{x}{4} = \frac{\sqrt{7}}{4}.$$

YOU TRY IT 24.2. Evaluate $\sin(\arctan 3x)$.

ANSWER TO YOU TRY IT 24.2. Draw a triangle whose tangent is $3x$.



$$(3x)^2 + 1^2 = z^2 \Rightarrow z = \sqrt{9x^2+1}.$$

So

$$\sin(\arctan 3x) = \sin(\theta) = \frac{3x}{z} = \frac{3x}{\sqrt{9x^2+1}}.$$

Derivatives of $\arcsin x$ *and* $\arctan x$

Surprisingly, it is relatively easy to determine the derivatives of the inverse trig functions, assuming that they are differentiable. We will use implicit differentiation (really just the chain rule in disguise) just as we did when we figured out the derivative of $\ln x$.

Let's first determine the derivative of $y = \arcsin x$ for $-1 \le x \le 1$. We want to find $\frac{dy}{dx}$. We start with

$$y = \arcsin x$$

Apply the inverse function (the sine function) to both sides:

$$\sin(y) = \sin(\arcsin x) = x$$

Take the derivative (use implicit differentiation on the left)

$$\frac{d}{dx}[\sin(y)] = \frac{d}{dx}[x]$$
$$\cos(y)\frac{dy}{dx} = 1$$

Solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\cos(\arcsin x)}$$

Example 24.1.4 we found that $\cos(\arcsin x) = \sqrt{1-x^2}$, so

$$\frac{dy}{dx} = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-x^2}}$$

That is

$$\boxed{\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}.}$$

The derivative of $y = \arctan x$ for $-\infty < x < \infty$ is determined in a similar fashion. We want to find $\frac{dy}{dx}$. Start with

$$y = \arctan x$$

Apply the inverse function (the tangent function) to both sides:

$$\tan(y) = \tan(\arctan x) = x$$

Take the derivative (use implicit differentiation on the left)

$$\frac{d}{dx}[\tan(y)] = \frac{d}{dx}[x]$$
$$\sec^2(y)\frac{dy}{dx} = 1$$

Solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)} = \frac{1}{\sec^2(\arctan x)}$$

In Example 24.1.5 we showed $\sec(\arctan x) = \sqrt{1+x^2}$, so $\sec^2(\arctan x) = 1 + x^2$. Therefore

$$\frac{dy}{dx} = \frac{1}{\sec^2(\arctan x)} = \frac{1}{1+x^2}$$

That is

$$\boxed{\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}.}$$

YOU TRY IT 24.3 (Extra Credit). Determine the formula for the derivative of $\arccos x$ using the method above. Show your work.

Keep going and find the derivatives of the remaining three inverse trig functions. Again show your work.

Chain Rule Versions

The chain rule versions of both derivative formulas are:

$$\boxed{\frac{d}{dx}(\arcsin u) = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}} \qquad \boxed{\frac{d}{dx}(\arctan u) = \frac{1}{1+u^2} \cdot \frac{du}{dx}}$$

EXAMPLE 24.1.7. Let's use these formulas to find the derivatives of the following:

$$\frac{d}{dx}(\arctan \overbrace{e^{3x}}^{u=e^{3x}}) = \frac{1}{1+u^2} \cdot \frac{du}{dx} = \frac{1}{\underbrace{1+(e^{3x})^2}_{1+u^2}} \cdot \overbrace{3e^{3x}}^{du/dx} = \frac{3e^{3x}}{1+e^{6x}}.$$

$$\frac{d}{dx}(\arcsin \overbrace{3x^2}^{u=3x^2}) = \frac{1}{\underbrace{\sqrt{1-(3x^2)^2}}_{\sqrt{1-u^2}}} \cdot \overbrace{6x}^{du/dx} = \frac{6x}{\sqrt{1-9x^4}}.$$

$$\frac{d}{dx}(e^{\arctan 3x}) = e^{\arctan 3x} \cdot \frac{1}{\underbrace{1+9x^2}_{1+u^2}} \cdot \overbrace{3}^{du/dx} = \frac{3e^{\arctan 3x}}{1+9x^2}.$$

$$\begin{aligned}\frac{d}{dx}(\sin 2x \arctan \overbrace{5x^2}^{u^2}) &= 2\cos 2x \arctan 5x^2 + \sin 2x \cdot \frac{1}{\underbrace{1+25x^4}_{1+u^2}} \cdot \overbrace{10x}^{du/dx} \\ &= 2\cos 2x \arctan 5x^2 + \frac{10x \sin 2x}{1+25x^4}.\end{aligned}$$

$$\frac{d}{dx}(\ln|\arcsin 3x|) = \frac{1}{\arcsin 3x} \cdot \frac{1}{\underbrace{\sqrt{1-9x^2}}_{\sqrt{1-u^2}}} \cdot \overbrace{3}^{du/dx} = \frac{3}{(\arcsin 3x)\sqrt{1-9x^2}}.$$

$$D_x(|\arcsin(\ln 3x)) = \frac{1}{\underbrace{\sqrt{1-[\ln(3x)]^2}}_{\sqrt{1-u^2}}} \cdot \underbrace{\frac{1}{3x} \cdot 3}_{du/dx} = \frac{1}{x\sqrt{1-[\ln(3x)]^2}}. \qquad (u = \ln(3x))$$

YOU TRY IT 24.4. Find the derivatives of these functions:

(*a*) $\frac{d}{dx}\arctan(6x^2)]$ (*b*) $\frac{d}{dx}[\arcsin(\sqrt{x})]$ (*c*) $\frac{d}{dx}[\arctan(e^{2x})]$ (*d*) $\frac{d}{dx}[\arcsin(\arcsin x)]$

(*e*) $\frac{d}{dx}[\arctan(\ln|6x|)]$ (*f*) $\frac{d}{dx}[\arcsin(6e^{\sin x})]$ (*g*) $\frac{d}{dx}(e^{2\arcsin x^2})$ (*h*) $\frac{d}{dx}[(\arcsin 2x)(\tan 5x^2)]$

(*i*) $\frac{d}{dx}(\ln|\arctan e^{x^4+1}|)$ (*j*) $D_x(2^{\arcsin 3x^2})$ (*k*) $D_x(\arctan x)^{\sin x}$

The answers are on the next page.

Answers.

1. Answers to YOU TRY IT 24.4 . The u's are for the inverse trig functions.

(a) $\dfrac{d}{dx}(\arctan(6x^2)) = \dfrac{1}{1+36x^4} \cdot 12x = \dfrac{12x}{1+36x^4}$ $\qquad (u = 6x^2)$

(b) $\dfrac{d}{dx}(\arcsin(\sqrt{x})) = \dfrac{1}{\sqrt{1-x}} \cdot \dfrac{1}{2} \cdot x^{-1/2} = \dfrac{1}{2\sqrt{x}\sqrt{1-x}}$ $\qquad (u = \sqrt{x})$

(c) $\dfrac{d}{dx}(\arctan(e^{2x})) = \dfrac{2e^{2x}}{1+e^{4x}}$ $\qquad (u = e^{2x})$

(d) $\dfrac{d}{dx}(\arcsin(\arcsin x)) = \dfrac{1}{\sqrt{1-(\arcsin x)^2}} \cdot \dfrac{1}{\sqrt{1-x^2}}$ $\qquad (u = \arcsin x))$

(e) $\dfrac{d}{dx}[\arctan(\ln|6x|)] = \dfrac{1}{1+(\ln|6x|)^2} \cdot \dfrac{1}{6x} \cdot 6 = \dfrac{1}{x[1+(\ln|6x|)^2]}$ $\qquad (u = \ln|6x|)$

(f) $\dfrac{d}{dx}(\arcsin(6e^{\sin x})) = \dfrac{1}{\sqrt{1-(6e^{\sin x})^2}} \cdot (6e^{\sin x})(\cos x) = \dfrac{6\cos x e^{\sin x}}{\sqrt{1-(6e^{\sin x})^2}}$ $\qquad (u = 6e^{\sin x})$

(g) $\dfrac{d}{dx}(e^{2\arcsin x^2}) = (e^{2\arcsin x^2}) \cdot 2 \cdot \dfrac{1}{\sqrt{1-(x^2)^2}} \cdot 2x = \dfrac{4xe^{2\arcsin x^2}}{\sqrt{1-x^4}}$ $\qquad (u = x^2)$

(h) $\dfrac{d}{dx}[\arcsin 2x(\tan 5x^2)] = \dfrac{2\tan 5x^2}{\sqrt{1-4x^2}} + (\arcsin 2x)10x\sec^2(5x^2)$ $\qquad (u = 2x,\ 5x^2)$

(i) $\dfrac{d}{dx}(\ln|\arctan e^{x^4+1}|) = \dfrac{1}{\arctan e^{x^4+1}} \cdot \dfrac{1}{1+(e^{x^4+1})^2} \cdot e^{x^4+1} \cdot 4x^3 = \dfrac{4x^3e^{x^4+1}}{(\arctan e^{x^4+1})(1+e^{2x^4+2})}$

(j) $D_x(2^{\arcsin 3x^2}) = 2^u \cdot \ln u \cdot \dfrac{du}{dx} = 2^{\arcsin 3x^2} \cdot \ln 2 \cdot \dfrac{1}{\sqrt{1-9x^4}} \cdot 6x = \dfrac{6x \cdot 2^{\arcsin 3x^2} \cdot \ln 2}{\sqrt{1-9x^4}}.$

(k) Use logarithmic differentiation.

$$\begin{aligned}
y &= (\arctan x)^{\sin x} \\
\ln y &= \ln(\arctan x)^{\sin x} \\
\ln y &= \sin x \ln(\arctan x) \\
\frac{1}{y}\frac{dy}{dx} &= \cos x \ln(\arctan x) + \sin x \cdot \frac{1}{1+x^2} \\
\frac{dy}{dx} &= y\left[\cos x \ln(\arctan x) + \frac{\sin x}{1+x^2}\right] \\
\frac{dy}{dx} &= (\arctan x)^{\sin x}\left[\cos x \ln(\arctan x) + \frac{\sin x}{1+x^2}\right]
\end{aligned}$$