*Working out the Critical Point and Closed Interval Theorems*

FROM THE GRAPHS we saw that sometimes we have extrema at critical points. The next theorem makes the connection explicit.

**DEFINITION.** Assume that $f$ is defined at $c$. Then $c$ is a **critical point of** $f$ if either

1. $f'(c)$ does not exist or
2. $f'(c) = 0$.

**THEOREM 27.1.1** (CPT: The Critical Point Theorem)**.** If $f$ has a local max or min at $c$, then $c$ is a critical point of $f$.

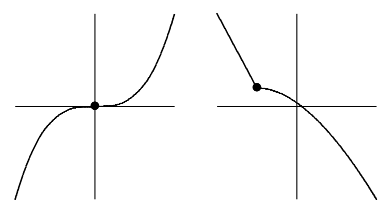
*Proof.* We prove the case where $f$ has a a local max at $c$. A similar proof works for a local min. There are two possibilities: Either $f$ is not differentiable at $c$ or it is.

1. If $f$ is not differentiable at $c$, then $f'(c)$ DNE and so $c$ is a ____________________.
2. If $f$ is differentiable, then we need to show $f'(c) = 0$ for it to be a critical point. $f$ has a local max at $c$ which means $f(x) \le f(c)$ for all $x$ near $c$ which means
$$\boxed{f(x) - f(c) \le 0} \quad \text{near } c.$$
Because $f$ is differentiable at $c$, the derivative
$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \quad \text{exists.}$$
So both one-sided limits (derivatives) exist and are equal at $c$. So
$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \rule{2cm}{0.4pt} \ge \rule{1cm}{0.4pt}$$
while
$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = \rule{2cm}{0.4pt} \le \rule{1cm}{0.4pt}.$$
Since both one-sided limits are equal, the only possibility is that they are both ____. In other words, $f'(c) = 0$ so $c$ is a ____________________.

$\square$

**YOU TRY IT 27.12.** Do the proof for the case when $f$ has a local min at $c$. What changes?

**Caution:** Notice that the CPT does NOT say that if $c$ is a critical point, then $f$ has a relative extrema at $c$. It is possible to have a $f'(c) = 0$ or DNE and not have an extreme point (see below).



*Recap: Putting Theorems Together.* Now assume that $f$ is continuous on the closed interval $[a, b]$. Then the EVT implies that $f$ has an **absolute max** at some point $c$ in $[a, b]$. There are two possibilities:

1. $c$ is one of the endpoints, $c = a$ or $c = b$
2. OR $c$ is between $a$ and $b$... so $a < c < b$. But then $c$ is not only an absolute max, it is also a **relative max** on $(a, b)$. So the CPT says that $c$ is a critical point of $f$.

The same is true for an absolute min of $f$ on $[a, b]$. Thus, we have proven

**THEOREM 27.1.2** (CIT: The Closed Interval Theorem)**.** Let $f$ be a continuous function on a closed interval $[a, b]$. Then the absolute extrema of $f$ occur either at critical points of $f$ on the open interval $(a, b)$ or at the endpoints $a$ and/or $b$.

*Algorithm (Recipe) for Finding Absolute Extrema.* To find the absolute extrema of a continuous function $f$ on a closed interval $[a, b]$

1. Find the critical points of $f$ on the open interval $(a, b)$ and evaluate $f$ at each such point.
2. Evaluate $f$ at each of the endpoints, $x = a$ and $x = b$.
3. Compare the values; the largest is the absolute max and the smallest is the absolute min.

*Working out the proof of Rolle's Theorem*

ROLLE'S THEOREM DEALS with functions that have the same starting and ending values.

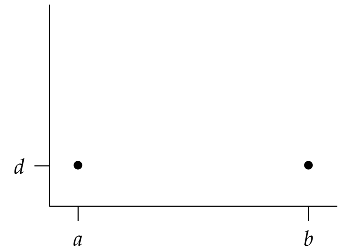


Figure 27.7: Draw a graph of a function starting and ending at the same height $d$ that satisfies the conditions of Rolle's Theorem.

**THEOREM 27.1.3** (Rolle's Theorem). Assume that

1. $f$ is continuous on the closed interval $[a, b]$;
2. $f$ is differentiable on the open interval $(a, b)$;
3. $f(a) = f(b)$, i.e., $f$ has the same value at both endpoints.

Then there's some point $c$ between $a$ and $b$ so that $f'(c) = 0$.

**EXAMPLE 27.1.4.** How does Rolle's theorem apply to tossing a ball up and catching it at the same height? What about about walking a long a straight line and starting and ending at the same point?

*Proof.* Since $f$ is continuous on the closed interval $[a, b]$, $f$ must have absolute maximum and minimum values on $[a, b]$ by the ______________________________ Theorem. There are two possibilities: either

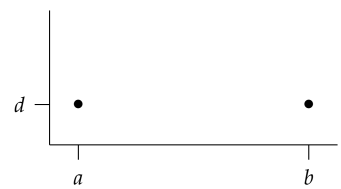


Figure 27.8: A graph for case (1) of Rolle's Theorem.

1. Both extreme values (max and min) occur at the endpoints of $[a, b]$ (draw such a function) or

   occur at the endpoints of $[a, b]$ (draw such a function) or

2. at least one of the extreme values occurs at a critical point of $f$.

1. In the first case $f(a) = f(b) = d$ is **both** the **minimum** value and the **maximum** value of $f$ on $[a, b]$. This can only happen if $f$ is constant on the interval. But if $f$ is constant, $f'(c) = 0$ for *any* point $c$ between $a$ and $b$.

2. In the second case, if $f$ has an extreme point $c$ between $a$ and $b$, since $f$ is differentiable, then by the ______________________________ Theorem $f'(c) =$ ________.

In either case there is a point $c$ where $f'(c) = 0$. □

**EXAMPLE 27.1.5.** Show how Rolle's theorem applies to $f(x) = x^2 - 5x$ on $[1, 4]$.

**Solution.** Check that $f(x)$ satisfies the three conditions of Rolle's Theorem.

1. $f$ is continuous on the closed interval $[1, 4]$ because $f(x)$ is a ______________________________
2. $f$ is differentiable on the open interval $(1, 4)$ because$f(x)$ is a ______________________________
3. $f(1) =$ ________ and $f(4) =$ ________, i.e., $f$ has the ______________ value at both endpoints.

The three conditions are satisfied so Rolle's Theorem applies.
So there is some point $c$ between 1 and 4 so that $f'(c) =$ ________.
But $f'(x) =$ ______________ $= 0$ at $x =$ ________. So $c =$ ________.

*Working out the proof of the Mean Value Theorem*

ROLLE'S THEOREM IS USED to prove the more general result, called the Mean Value theorem. **You should be able to state this theorem and draw a graph that illustrates it.**

**THEOREM 27.6.6** (MVT: The Mean Value Theorem). Assume that

1. $f$ is continuous on the closed interval $[a, b]$;
2. $f$ is differentiable on the open interval $(a, b)$;

Then there is some point $c$ in $(a, b)$ so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

This is the same as saying: $f(b) - f(a) = f'(c)(b - a)$.

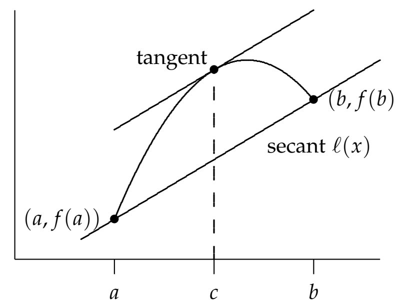



Figure 27.9: Under the conditions of the MVT, there's a point $c$ in $(a, b)$ so that the secant slope $f'(c) = \frac{f(b)-f(a)}{b-a}$ equals the tangent slope $f'(c)$.

*Proof.* Strategy: Modify $f$ so that we can apply Rolle's theorem.

Let $\ell(x)$ be the (secant) line (to $f$) that passes through the points $(a, f(a))$ and $(b, f(b))$. Even though we have not figured out the equation for $\ell(x)$ we still know its derivative:

$$\ell'(x) = \text{slope of the line } \ell = \frac{\Delta y}{\Delta x} = \underline{\hspace{6cm}}. \quad (27.2)$$

Since $\ell(x)$ is differentiable everywhere it is also ________________________.

Now consider the difference function $g(x) = f(x) - \ell(x)$. Since $f$ and $\ell$ are both continuous on $[a, b]$, then $g$ is also ________________________. Since $f$ and $\ell$ are both differentiable on $(a, b)$, then $g$ is also ________________________.

Now check the values of $g$ at the endpoints $a$ and $b$ (look at the graph):

$$g(a) = f(a) - \ell(a) = \underline{\hspace{2.5cm}} \text{ and } g(b) = f(b) - \ell(b) = \underline{\hspace{2.5cm}}.$$

So Rolle's theorem applies to $g$. This means there is a point $c$ between $a$ and $b$ such that $g'(c) =$ ____________. But then $g'(c) = f'(c) - \ell'(c) = 0$ which means using Equation (27.2)

$$f'(c) = \ell'(c) = \underline{\hspace{6cm}}.$$

□

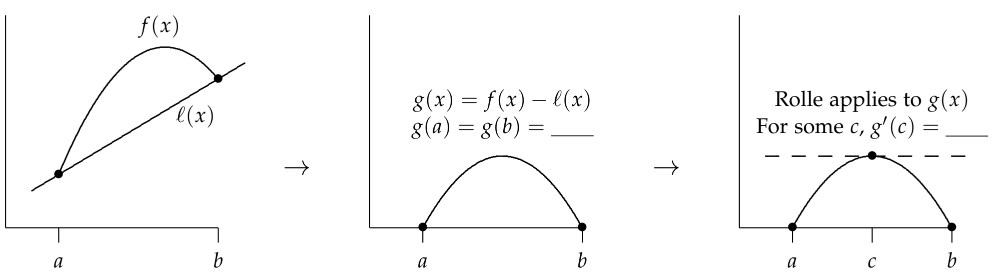



Figure 27.10: **Geometric recap of the proof of the MVT.** $f(x)$ and $\ell(x)$ have the same value at the endpoints $a$ and $b$ so their difference is ____ at both $a$ and $b$. Consequently Rolle's theorem applies to $g$ and there is a point $c$ between $a$ and $b$ such that $g'(c) =$ ____.

**EXAMPLE 27.6.7.** Show how the MVT theorem applies to $f(x) = x^2$ on $[1, 3]$.

**Solution.** Check that $f(x)$ satisfies the two conditions of the MVT.

1. $f$ is continuous on the closed interval $[1, 3]$ because $f(x)$ is a ____________________________
2. $f$ is differentiable on the open interval $(1, 3)$ because$f(x)$ is a ____________________________

The two conditions are satisfied so the MVT applies.

So there is some point $c$ between 1 and 3 so that $f'(c) = \frac{f(3)-f(1)}{3-1} =$ ________.

But $f'(x) = 2x$, so $c =$ ________.

## *Using the MVT: Increasing and Decreasing Functions*

The MVT is valuable because it is used to prove lots of other calculus results. First let's be clear on what increasing and decreasing functions are.

**DEFINITION 27.6.8.** Assume $f$ is defined on an interval $I$. $f$ is **increasing on** $I$ if whenever $a$ and $b$ are in $I$ and $a < b$, then $f(a) < f(b)$. Likewise, $f$ is **decreasing on** $I$ if whenever $a$ and $b$ are in $I$ and $a < b$, then ______________.

Figure 27.11: Draw an increasing differentiable function $f(x)$. What can you say about its slope $f'(x)$?

**EXAMPLE 27.6.9.** Draw an increasing differentiable function $f(x)$ in the margin. What can you say about its slope? Consequently the following theorem should not be surprising.

**THEOREM 27.6.10 (Increasing/Decreasing Test).** 1. If $f'(x) > 0$ for all $x$ in an interval $I$, then $f$ is increasing on $I$.

2. If $f'(x) < 0$ for all $x$ in an interval $I$, then $f$ is decreasing on $I$.

*Proof.* We will use the MVT to prove this. We'll prove (2). Try proving (1) as an Extra Credit problem.

In case (2) we assume $f'(x)$______________ for all $x$ in $I$.

Let $a$ and $b$ be any two points in $I$ with $a < b$. Using the Definition 27.6.8, to show that $f$ is decreasing, we need to show that ______________.

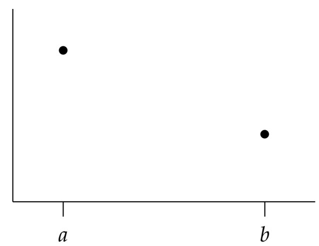



Figure 27.12: Here's what we are trying to prove in case (2): That $f(b) < f(a)$.

But $f$ is differentiable on $I$ so $f$ is ____________________________ on $[a, b]$, and of course, $f$ differentiable on $(a, b)$. So the MVT applies to $f$ on $[a, b]$. So there's a point $c$ in ______________ so that

$$f'(c) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.$$

This means

$$f(b) - f(a) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.$$

From the assumption in case (2), we know that $f'(c)$______________ and $(b - a)$ ______________. So

$$f(b) - f(a) = \overbrace{f'(c)}^{\_\_\_\_} \cdot \overbrace{(b-a)}^{\_\_\_\_} < \_\_\_\_\_\_.$$

If $f(b) - f(a) < 0$, then $f(b) <$ ______________, so $f$ is _____________________. □

**YOU TRY IT 27.13.** Prove case (1) of the Increasing/Decreasing Test where $f'(x) > 0$.

**EXAMPLE 27.6.11.** Let $f(x) = x^4 - 6x^2 + 1$. Where is $f$ increasing? Decreasing? Where does it have relative extrema? What theorem do you use to answer these questions?