## Optimization and Calculus

## Introduction to Extrema and Optimization

Our goal for the next several days is to apply calculus theory to the practical problem of solving optimization problems. This is one of the most important applications of elementary differential calculus and it is widely used.

EXAMPLE 26.1.1. A soup can needs to be designed to hold 18 oz . What dimensions for the radius and height of the can minimize the cost (i.e., the materials) for the can. Here, optimal means minimize.


EXAMPLE 26.1.2. Your Favorite Band is playing at the Smith Opera House. What ticket price will maximize their revenue? Here, optimal means maximize.

EXAMPLE 26.1.3. For several years I served on a committee that investigated what tuition will maximize the revenue for Hobart \& William Smith Colleges.

In all of these situations, we wish to locate the maximum or minimum values of a function efficiently using differential calculus. Such points are called extreme values of the function.

## Extrema Terminology

There are several types of extreme values and it is important that we define them carefully.

DEFINITION 26.1.4. Let $f$ be a function defined on an interval $I$ containing the point $c$.
(a) $f$ has an absolute (global) maximum at $c$ if $f(c) \geq f(x)$ for all $x$ in $I$. The number $f(c)$ is the maximum value of $f$.
(b) $f$ has an absolute (global) minimum at $c$ if $f(c) \leq f(x)$ for all $x$ in $I$. The number $f(c)$ is the minimum value of $f$.
(c) Absolute maxima or minima are called absolute extreme values (extrema) of $f$

The absolute extrema are marked with $\star$ in these graphs. Notice that the second graph has no absolute max or min (since it does not contain the endpoints) and that the third graph has no absolute min (the small dot is part of the graph but is
not a min) and two maxima (at the endpoints.




DEFINITION 26.1.5. Note open intervals!
(a) If $f(c) \geq f(x)$ for all $x$ in some open interval containing $c$, then $f(c)$ is a relative (local) maximum value of $f$. (Or: $f$ has a local max at $c$.)
(b) If $f(c) \leq f(x)$ for all $x$ in some open interval containing $c$, then $f(c)$ is a relative (local) minimum value of $f$. (Or: $f$ has a local min at $c$.)

In the graphs above, the first has three local minima and two local maxima (note that the endpoints are not local extrema because there is no open interval containing the endpoints. The second and third graphs have no local extrema.

We can see in third graph that the function would have both a local and global minimum value if the removable discontinuity were eliminated, that is, if it were continuous. So we see that continuity is important in finding extreme values. On the other hand, the second graph is continuous but does not have any relative or global extrema. This time the problem is that the interval over which it is defined is not closed. If the interval were closed and the endpoints of the graph included, then it would have at least had a global max and a global min, though it still would not have had any local extrema.

The second fact about continuous functions on closed intervals is that they always have global extrema (though they may not have local ones).

## The interval matters

EXAMPLE 26.1.6. Left: The function $y=f(x)=x^{2}$ on the closed interval $[-1,2]$ has both an absolute max and min.

Middle: The function $y=f(x)=x^{2}$ on the open interval $(-1,2)$ has an absolute min but no absolute max.

Right: The function $y=f(x)=x^{2}$ on the open interval $(-0,2)$ has no absolute min or max.




## The type of function matters

The example above shows that a continuous function on an non-closed interval may not have an absolute max or min. When the interval is closed, if the function is not continuous, it may still not have have both an absolute max or min.

EXAMPLE 26.1.7. Left: A discontinuous function $y=f(x)$ on the closed interval $[0,3]$ that has an absolute max but no absolute min. Notice $f$ is defined at each point in the interval.

Middle: A discontinuous function $y=f(x)$ on the closed interval $[0,3]$ that has an absolute max but no absolute min. Notice $f$ is defined at each point in the interval.

Right: A discontinuous function $y=f(x)$ on the closed interval $[0,3]$ that has both an absolute max and absolute min. Notice $f$ is defined at each point in the interval.


It turns out that when we have a continuous function on a closed interval, life is good!

THEOREM 26.1.8 (EVT: The Extreme Value Theorem). A function $f$ that is continuous on a closed interval $[a, b]$ has an absolute maximum value and an absolute minimum value in that interval $[a, b]$.

YOU TRY IT 26.1. Mark the global maximum and minimum values of the continuous function on the closed interval $[a, b]$ on the figure in the margin.

YOU TRY IT 26.2. (From class). Draw a function that satisfies the given conditions or explain why this is impossible.
(a) A continuous function on $(1,8)$ which has no absolute minimum
(b) A continuous function on $[1,8]$ which has no absolute extreme points.
(c) A function on $[1,8]$ which has no absolute maximum.
(d) A continuous function on $[1,8]$ for which $f(1)=-3$ and $f(8)=4$ and which is never 0 (has no roots).
(e) A continuous function on $(1,8)$ for which $f(3)$ is a relative max and $f(5)$ is a relative min but for which $f$ has no absolute max or min.

YOU TRY IT 26.3. (More general). Draw your own graph that shows that the conclusion of the EVT can fail-that is, the function may not have a global max or a global min-if the function is not continuous even if it is defined at every point on a closed interval.

Draw another function hat shows that the conclusion of the EVT can fail on a non-closed interval even if the function is continuous.

These examples show that for a function to always have both a global max and a global min, it must be continuous on a closed interval. Both conditions are important.

The IVT and EVT are two great theorems (and both are very hard to prove; take Math 331). The latter, in particular, tells us that can always optimize a continuous function on a closed interval, which is what motivated us in the first place. However, the EVT does not tell us HOW to find the extrema, it only tells that they exist. When a function is differentiable on a closed interval $[a, b]$, we will see that we can give an algorithm (recipe) for how to find the extrema.

EXAMPLE 26.1.9 (A Look Ahead). The function to the right is differentiable, hence continuous on the closed interval $[0,10.5]$. The EVT says the function should have both an absolute max and an absolute min. Mark them. For a differentiable function where should we look for extreme values? Mark at all the local extrema. What is the common property that all these local extrema share?


## Optimization: Differentiability and Extrema

Our first theorem tells us that when $f$ is differentiable, then at a local high or low point, the function has a horizontal tangent. In fact, if we first define a new term, then we can say even more.

DEFINITION 29.1.1. Assume that $f$ is defined at $c$. Then $c$ is a critical point of $f$ if either

1. $f^{\prime}(c)=0$ or
2. $f^{\prime}(c)$ does not exist.

EXAMPLE 29.1.2. Find the critical points of these four graphs


EXAMPLE 29.1.3. Find the critical points of $f(x)=3 x^{2 / 3}-2 x$.
Solution. We need to see where $f^{\prime}(x)=0$ or DNE.

$$
f^{\prime}(x)=2 x^{-1 / 3}-2=\frac{2}{x^{1 / 3}}-2=0 \Longleftrightarrow \frac{1}{x^{1 / 3}}=1 \Longleftrightarrow 1=x^{1 / 3} \Longleftrightarrow 1=x
$$

Also notice

$$
f^{\prime}(x)=2 x^{-1 / 3}-2=\frac{2}{x^{1 / 3}}-2 \mathrm{DNE} \Longleftrightarrow x=0
$$

So $f$ has two critical points, $x=0,1$.
EXAMPLE 29.1.4. Find the critical points of $f(x)=\arctan \left(e^{2 x}-4 x\right)$.
Solution. We need to see where $f^{\prime}(x)=0$ or DNE.
$f^{\prime}(x)=\frac{2 e^{2 x}-4}{1+\left(e^{2 x}-4 x\right)^{2}}=0 \Longleftrightarrow 2 e^{2 x}-4=0 \Longleftrightarrow e^{2 x}=2 \Longleftrightarrow 2 x=\ln 2 \Longleftrightarrow x=\frac{\ln 2}{2}$.
Notice $f^{\prime}(x)$ always exists, so there is only one critical point: $x=\frac{\ln 2}{2}$.
From the graphs we saw that sometimes we have extrema at critical points. The next theorem makes the connection explicit.

THEOREM 29.1.5 (CPT: The Critical Point Theorem). If $f$ has a local max or min at $c$, then $c$ is a critical point of $f$.

Proof. We will examine the case where $f$ has a local max at $c$. A similar proof works for a local min. There are two possibilities: Either $f$ is not differentiable at $c$ or it is.

1. If $f$ is not differentiable at $c$, then $c$ is automatically a critical point by definition.
2. If $f$ is differentiable, then we need to show $f^{\prime}(c)=0$ for it to be a critical point.

Now $f$ has a local max at $c$ means $f(x) \leq f(c)$ for all $x$ near $c$ which means that $f(x)-f(c) \leq 0$ near $c$. Because $f$ is differentiable at $c, f^{\prime}(c)$ exists, that is,

$$
f^{\prime}(c)=\lim _{x \rightarrow c} \frac{f(x)-f(c)}{x-c} \quad \text { exists. }
$$

Consequently, both one-sided limits (derivatives) exist and are equal at $c$. Notice that the numerator $f(x)-f(c) \leq 0$ so

$$
f^{\prime}(c)=\lim _{x \rightarrow c^{-}} \frac{f(x)-f(c)}{x-c}=\frac{- \text { or } 0}{-} \geq 0
$$

while

$$
f^{\prime}(c)=\lim _{x \rightarrow c^{+}} \frac{f(x)-f(c)}{x-c}=\frac{- \text { or } 0}{+} \leq 0 .
$$

Since both one-sided limits are equal, the only possibility is that they are both $o$. In other words, $f^{\prime}(c)=0$ so $c$ is a critical point.

YOU TRY IT 29.4. Do the proof for the case when $f$ has a local min at $c$. What changes?

Caution: Notice that the CPT does NOT say that if $c$ is a critical point, then $f$ has a relative extrema at $c$. It is possible to have a zero derivative and not have an extreme point. It just says among all the critical points we will find the relative extrema.




Putting Theorems Together. Now assume that $f$ is continuous on the closed interval $[a, b]$. Then the EVT implies that $f$ has an absolute max at some point $c$ in $[a, b]$.
There are two possibilities:

1. $c$ is one of the endpoints, $c=a$ or $c=b$
2. OR $c$ is between $a$ and $b \ldots$ so $a<c<b$. But then $c$ is not only an absolute max, it is also a relative max on $(a, b)$. So the CPT implies that $c$ is a critical point of $f$.

The same is true for an absolute min of $f$ on $[a, b]$. Thus, we have proven
THEOREM 29.1.6 (CIT: The Closed Interval Theorem). Let $f$ be a continuous function on a closed interval $[a, b]$. Then the absolute extrema of $f$ occur either at critical points of $f$ on the open interval $(a, b)$ or at the endpoints $a$ and/or $b$.

Figure 29.1: Three critical points which are not relative extreme values.

From this we get:

Algorithm (Recipe) for Finding Absolute Extrema. To find the absolute extrema of a continuous function $f$ on a closed interval $[a, b]$

1. Find the critical points of $f$ on the open interval $(a, b)$ and evaluate $f$ at each such point.
2. Evaluate $f$ at each of the endpoints, $x=a$ and $x=b$.
3. Compare the values; the largest is the absolute max and the smallest is the absolute min.

EXAMPLE 29.1.7. Find the absolute extrema of $p(x)=\frac{1}{3} x^{3}-2 x^{2}+3 x+1$ on $[0,2]$.
Solution. Notice that $p$ is a polynomial so it is continuous and the interval is closed. So we can apply the CIT algorithm.

1. Find the critical points of $p$ on the open interval $(0,2)$ and evaluate $p$ at each such point.

$$
p^{\prime}(x)=x^{2}-4 x+3=(x-3)(x-1)=0 ; \quad \text { at } x=1,3 .
$$

Notice that $x=3$ lies outside the interval, so it is eliminated. $x=1$ is the only critical point in $(0,2)$. Finally, $p(1)=\frac{1}{3}-2+3+1=\frac{7}{3}$.
2. Evaluate $f$ at each of the endpoints, $x=0$ and $x=2: p(0)=1$ and $p(2)=\frac{8}{3}-8+6+1=\frac{5}{3}$.
3. The absolute max is $\frac{7}{3}$ at $x=1$ and the absolute $\min$ is 1 at $x=0$.

EXAMPLE 29.1.8. Find the absolute extrema of $f(x)=3 x^{2 / 3}$ on $[-1,8]$.
Solution. Notice that $f$ is a root function so it is continuous and the interval is closed. So we can apply the CIT algorithm.

1. Find the critical points of $f$ on $(-1,8)$ :

$$
f^{\prime}(x)=2 x^{-1 / 3}=\frac{2}{x^{1 / 3}} \neq 0 .
$$

However, $f^{\prime}(x)$ DNE at $x=0$. So $x=0$ is a critical point in the interval. And $f(0)=0$.
2. Evaluate $f$ at the endpoints, $x=-1$ and $x=8: f(-1)=-3$ and $f(8)=4$.
3. The absolute max is 4 at $x=2$ and the absolute $\min$ is o at $x=0$.

EXAMPLE 29.1.9. Find the absolute extrema of $f(x)=x^{2} e^{-x^{2} / 4}$ on $[-1,4]$.
Solution. Notice that $f$ is a the product of an exponential and a polynomial so it is continuous and the interval is closed. So we can apply the CIT algorithm.

1. Find the critical points of $f$ on $(-1,4)$ :
$f^{\prime}(x)=2 x e^{-x^{2} / 4}+x^{2} e^{-x^{2} / 4}\left(-\frac{2 x}{4}\right)=2 x e^{-x^{2} / 4}-\frac{1}{2} x^{3} e^{-x^{2} / 4}=e^{-x^{2} / 4}\left(2 x-\frac{1}{2} x^{3}\right)=0$.
So $2 x-\frac{1}{2} x^{3}=x\left(2-\frac{1}{2} x^{2}\right)=0=0$ at $x=0$ or $2=\frac{1}{2} x^{2} \Rightarrow 4=x^{2} ; x= \pm 2$. So $x=0,2$ are the critical points in the interval. And $f(0)=0$ while $f(2)=4 e^{-1} \approx 1.471$.
2. Evaluate $f$ at the endpoints, $x=-1$ and $x=4$ : $f(-1)=e^{-1 / 4} \approx 0.7788$ and $f(4)=16 e^{-4} \approx 0.293$.
3. The absolute max is $4 e^{-1}$ at $x=2$ and the absolute $\min$ is 0 at $x=0$.

YOU TRY IT 29.5. Find the absolute extrema of $f(x)=\ln \left(x^{2}+1\right)$ on $[-2,1]$. [Answer: Global $\max$ of $\ln 5$ at $x=-2$ and global min of 0 at $x=0$.]

YOU TRY IT 29.6. Find the absolute extrema of $f(x)=\sqrt{8 x-x^{2}}$ on [0,6]. [Answer: Global $\max$ of 4 at $x=4$ and global min of 0 at $x=0$.]

YOU TRY IT 29.7. Find the absolute extrema of $f(x)=x e^{x}$ on $[-2,3$ ]. [Answer: Global max of $3 e^{3}$ at $x=3$ and global min of $-e-1$ at $x=-1$.]

EXAMPLE 29.1.10. Two non-negative numbers $x$ and $y$ sum to 10 . What choices for $x$ and $y$ maximize $e^{x y}$ ?

Solution. We want to maximize $e^{x y}$. The function has two variables. If we let $y=10-x$ then we want to find the absolute max of $f(x)=e^{x(10-x)}=e^{10 x-x^{2}}$. Notice the possible values for $x$ are 0 to 10 , inclusive (since both $x$ and $y$ must be non-negative and since they sum to 10 ). So the interval is $[0,10]$ and is closed. Since $f$ is continuous we can apply the CIT algorithm.

1. Find the critical points of $f$ on $(0,10)$ :

$$
f^{\prime}(x)=(10-2 x) e^{10 x-x^{2}}=0 \text { at } x=5 .
$$

So $x=5$ is the only critical point in the interval. And $f(5)=e^{25}$.
2. Evaluate $f$ at the endpoints, $x=0$ and $x=10: f(0)=e^{0}=1$ and $f(10)=e^{0}=1$.
3. The absolute max is $e^{25}$ at $x=5$.

