## The Mean Value Theorem

## Introduction

דoday we discuss one of the most important theorems in calculus-the MVT. It says something about the slope of a function on a closed interval based on the values of the function at the two endpoints of the interval. It relates local behavior of the function to its global behavior. This theorem turns out to be the key to many other theorems about the graphs of functions and their behavior. We begin with a simple case.

## Rolle's Theorem

Rolle's theorem deals with functions that have the same starting and ending values.

THEOREM 30.1.1 (Rolle's Theorem). Assume that

1. $f$ is continuous on the closed interval $[a, b]$;
2. $f$ is differentiable on the open interval $(a, b)$;
3. $f(a)=f(b)$, i.e., $f$ has the same value at both endpoints.

Then there's some point $c$ between $a$ and $b$ so that $f^{\prime}(c)=0$.


EXAMPLE 30.1.2. How does Rolle's theorem apply to tossing a ball up and catching it at the same height? What about about walking a long a straight line and starting and ending at the same point?

Proof. By the CIT, $f$ must have maximum and minimum values on $[a, b]$. There are two possibilities: either (i) both of these extreme values occur at the endpoints of $[a, b]$ or (ii) at least one of the extreme values occurs at a critical point of $f$.

In case (i), $f(a)=f(b)=d$ is both the minimum value and the maximum value of $f$ on $[a, b]$. This can only happen if $f$ is constant on the interval. But if $f$ is constant, $f^{\prime}(c)=0$ for any point $c$ between $a$ and $b$. The situation in case (ii) is even easier. If $f$ has an extreme point $c$ between $a$ and $b$, since $f$ is differentiable, then by the CPT $f^{\prime}(c)=0$.

EXAMPLE 30.1.3. Show how Rolle's theorem applies to $f(x)=x^{2}-5 x$ on $[1,4]$.

Figure 30.2: When the hypotheses of Rolle's theorem are satisfied, there is a horizontal tangent, i.e., a critical point exists.

Solution. Check the three conditions

1. $f$ is continuous on the closed interval $[1,4]$ because it is a polynomial;
$f$ is differentiable on the open interval $(1,4)$ again because it is a polynomial;
$f(1)=-4$ and $f(4)=-4$, i.e., $f$ has the same value at both endpoints.
So there is some point $c$ between 1 and 4 so that $f^{\prime}(c)=0$. But $f^{\prime}(x)=2 x-5=0$ at $x=2.5$. This, then is the value of $c$.

## The Mean Value Theorem

Rolle's Theorem is used to prove the more general result, called the Mean Value theorem. You should be able to state this theorem and draw a graph that illustrates it.

THEOREM 30.1.4 (MVT: The Mean Value Theorem). Assume that

1. $f$ is continuous on the closed interval $[a, b]$;
2. $f$ is differentiable on the open interval $(a, b)$;

Then there is some point $c$ in $(a, b)$ so that

$$
f^{\prime}(c)=\frac{f(b)-f(a)}{b-a}
$$

This is equivalent to saying $f(b)-f(a)=f^{\prime}(c)(b-a)$.

Note: When $f(a)=f(b)$ we are back to Rolle's theorem. . . the conclusion of the MVT says in this case that $f^{\prime}(c)=0$.

Proof. Strategy: Modify $f$ so that we can apply Rolle's theorem.
Let $\ell(x)$ be the (secant) line (to $f$ ) that passes through the points ( $a, f(a)$ ) and $(b, f(b))$. Notice $\ell(x)$ is both continuous and differentiable everywhere (since it is a line). In fact, we know that $\ell^{\prime}(x)$ is just the slope of the line which is (always)

$$
\begin{equation*}
\ell^{\prime}(x)=\frac{f(b)-f(a)}{b-a} . \tag{30.1}
\end{equation*}
$$

Now consider the difference function $g(x)=f(x)-\ell(x)$. Since $f$ and $\ell$ are both continuous on $[a, b]$ so is $g$ and since $f$ and $\ell$ are both differentiable on $(a, b)$ so is g.



Further

$$
g(a)=f(a)-\ell(a)=f(a)-f(a)=0 \text { and } g(b)=f(b)-\ell(b)=f(b)-f(b)=0 .
$$



Figure 30.3: Under the conditions of the MVT, there's a point $c$ in $(a, b)$ so that the secant slope $f^{\prime}(c)=\frac{f(b)-f(a)}{b-a}$ equals the tangent slope $f^{\prime}(c)$.

Figure 30.4: $f(x)$ and $\ell(x)$ have the same value at the endpoints $a$ and $b$ so their difference is 0 at both $a$ and $b$. Consequently Rolle's theorem applies to $g$ and there is a point $c$ between $a$ and $b$ such that $g^{\prime}(c)=0$.

So Rolle's theorem applies to $g$. This means there is a point $c$ between $a$ and $b$ such that $g^{\prime}(c)=0$. But $g^{\prime}(c)=f^{\prime}(c)-\ell^{\prime}(c)=0$ which means using (30.1)

$$
f^{\prime}(c)=\ell^{\prime}(c)=\frac{f(b)-f(a)}{b-a}
$$

Mostly the MVT gets used to prove other theorems. But we can look at an example or two to see how it works.
EXAMPLE 30.1.5. Show how the MVT applies to $f(x)=x^{3}-6 x+1$ on $[0,3]$.
Solution. Check the two conditions (hypotheses)

1. $f$ is continuous on the closed interval $[0,3]$ because it is a polynomial;
$f$ is differentiable on the open interval $(0,3)$ again because it is a polynomial;

So the MVT applies: There is some point $c$ between $o$ and 3 so that

$$
f^{\prime}(c)=\frac{f(3)-f(0)}{3-0}=\frac{10-1}{3}=3
$$

Now

$$
\begin{aligned}
f^{\prime}(x)=3 x^{2}-6 & =3 \\
3 x^{2} & =9 \\
x & = \pm \sqrt{3}
\end{aligned}
$$

Only $x=\sqrt{3}$ is in the interval, so this is the value of $c$.
EXAMPLE 30.1.6. Show there does not exist a differentiable function on $[1,5]$ with $f(1)=-3$ and $f(5)=9$ with $f^{\prime}(x) \leq 2$ for all $x$.

Solution. The MVT would apply to such a function $f$ : So there is some point $c$ between 1 and 5 so that

$$
f^{\prime}(c)=\frac{f(5)-f(1)}{5-1}=\frac{9-(-3)}{4}=3
$$

But supposedly $f^{\prime}(x) \leq 2$ for all $x$. Contradiction. So no such $f$ can exist.

## Another Proof of the MVT

COROLLARY 30.1.7 (Corollary to Rolle's Theorem). Let $f$ and $g$ be two functions such that

1. $f$ and $g$ are continuous on the closed interval $[a, b]$;
2. $f$ and $g$ are differentiable on the open interval $(a, b)$;
3. $f(a)=g(a)$ and $f(b)=g(b)$, i.e., $f$ and $g$ have the same values at the endpoints.

Then there's some point $c$ between $a$ and $b$ so that $f^{\prime}(c)=g^{\prime}(c)$.

Proof. Consider the function $h(x)=f(x)-g(x)$. Then

1. $h$ is continuous on the closed interval $[a, b]$ since $f$ and $g$ are;
2. $h$ is differentiable on the open interval $(a, b)$ since $f$ and $g$ are;
3. Because $f(a)=g(a)$ and $f(b)=g(b)$, then $h(a)=f(a)-f(b)=0$ and similarly $h(b)=0$,so $h$ has the same value at both endpoints

By Rolle's theorem, there's some point $c$ between $a$ and $b$ so that $h^{\prime}(c)=0$. But then $0=h^{\prime}(c)=f^{\prime}(c)-g^{\prime}(c)$, so $f^{\prime}(c)=g^{\prime}(c)$.

THEOREM 30.1.8 (MVT: The Mean Value Theorem). Assume that

1. $f$ is continuous on the closed interval $[a, b]$;
2. $f$ is differentiable on the open interval $(a, b)$;

Then there is some point $c$ in $(a, b)$ so that

$$
f^{\prime}(c)=\frac{f(b)-f(a)}{b-a}
$$

This is equivalent to saying $f(b)-f(a)=f^{\prime}(c)(b-a)$.

Proof. Let $\ell(x)$ be the (secant) line (to $f$ ) that passes through the points $(a, f(a))$ and $(b, f(b))$. Notice $\ell(x)$ is both continuous and differentiable everywhere (since it is a line). In fact, we know that $\ell^{\prime}(x)$ is just the slope of the line which is (always)

$$
\begin{equation*}
\ell^{\prime}(x)=\frac{f(b)-f(a)}{b-a} \tag{30.2}
\end{equation*}
$$

Then $f(x)$ and $\ell(x)$ satisfy the conditions of Corollary 30.1.7. So there is a point $c$ between $a$ and $b$ so that

$$
f^{\prime}(c)=l^{\prime}(c) \stackrel{(30.2)}{=} \frac{f(b)-f(a)}{b-a}
$$

