The Mean Value Theorem

Introduction

ODAY WE DISCUSS one of the most important theorems in calculus—the MVT. It says something about the *slope* of a function on a closed interval based on the values of the function at the two endpoints of the interval. It relates *local* behavior of the function to its *global* behavior. This theorem turns out to be the key to many other theorems about the graphs of functions and their behavior. We begin with a simple case.

Rolle's Theorem

Rolle's theorem deals with functions that have the same starting and ending values.

THEOREM 30.1.1 (Rolle's Theorem). Assume that

1. f is continuous on the closed interval $[a, b]$;

2. f is differentiable on the open interval (a, b);

3. $f(a) = f(b)$, i.e., f has the same value at both endpoints.

Then there's some point c between a and b so that $f'(c) = 0$.
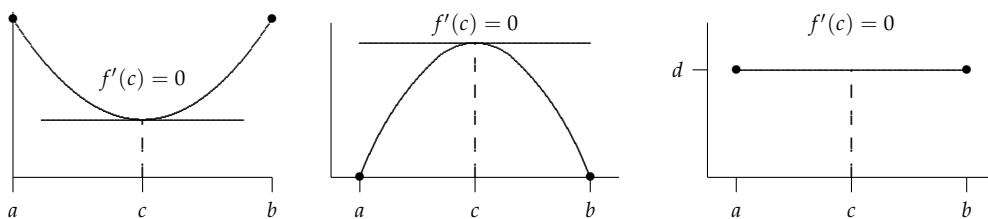

Figure 30.2: When the hypotheses of Rolle's theorem are satisfied, there is a horizontal tangent, i.e., a critical point exists.

EXAMPLE 30.1.2. How does Rolle's theorem apply to tossing a ball up and catching it at the same height? What about about walking a long a straight line and starting and ending at the same point?

Proof. By the CIT, f must have maximum and minimum values on $[a, b]$. There are two possibilities: either (i) both of these extreme values occur at the endpoints of $[a, b]$ or (ii) at least one of the extreme values occurs at a critical point of f.

In case (i), $f(a) = f(b) = d$ is both the minimum value and the maximum value of f on $[a, b]$. This can only happen if f is constant on the interval. But if f is constant, $f'(c) = 0$ for *any* point c between a and b. The situation in case (ii) is even easier. If f has an extreme point c between a and b, since f is differentiable, then by the CPT $f'(c) = 0$. $\square$

EXAMPLE 30.1.3. Show how Rolle's theorem applies to $f(x) = x^2 - 5x$ on $[1, 4]$.

Solution.   Check the three conditions

1. f is continuous on the closed interval $[1, 4]$ because it is a polynomial;

   f is differentiable on the open interval $(1, 4)$ again because it is a polynomial;

   $f(1) = -4$ and $f(4) = -4$, i.e., f has the same value at both endpoints.

So there is some point c between 1 and 4 so that $f'(c) = 0$. But $f'(x) = 2x - 5 = 0$ at $x = 2.5$. This, then is the value of c.

The Mean Value Theorem

Rolle's Theorem is used to prove the more general result, called the Mean Value theorem. **You should be able to state this theorem and draw a graph that illustrates it.**

THEOREM 30.1.4 (MVT: The Mean Value Theorem). Assume that

1. f is continuous on the closed interval $[a, b]$;

2. f is differentiable on the open interval (a, b);

Then there is some point c in (a, b) so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

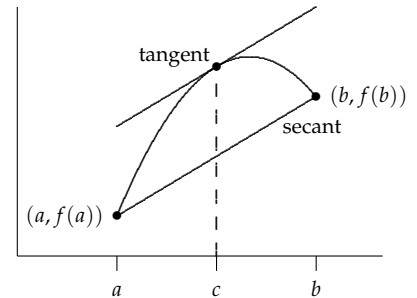This is equivalent to saying $f(b) - f(a) = f'(c)(b - a)$.

Figure 30.3: Under the conditions of the MVT, there's a point c in (a, b) so that the secant slope $f'(c) = \frac{f(b) - f(a)}{b - a}$ equals the tangent slope $f'(c)$.

Note: When $f(a) = f(b)$ we are back to Rolle's theorem... the conclusion of the MVT says in this case that $f'(c) = 0$.

Proof. Strategy: Modify f so that we can apply Rolle's theorem.

Let $\ell(x)$ be the (secant) line (to f) that passes through the points $(a, f(a))$ and $(b, f(b))$. Notice $\ell(x)$ is both continuous and differentiable everywhere (since it is a line). In fact, we know that $\ell'(x)$ is just the slope of the line which is (always)

$$\ell'(x) = \frac{f(b) - f(a)}{b - a}. \tag{30.1}$$

Now consider the difference function $g(x) = f(x) - \ell(x)$. Since f and ℓ are both continuous on $[a, b]$ so is g and since f and ℓ are both differentiable on (a, b) so is g.
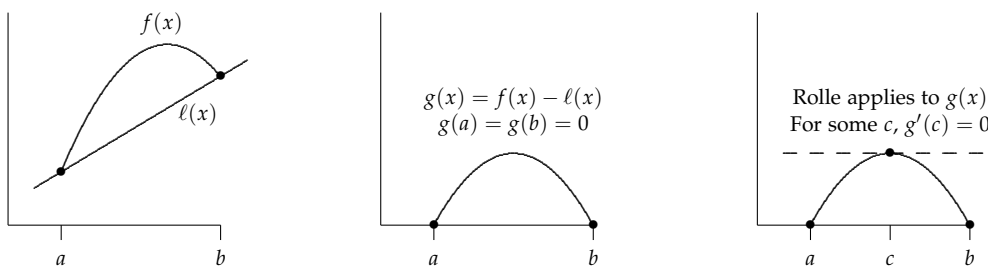

Figure 30.4: $f(x)$ and $\ell(x)$ have the same value at the endpoints a and b so their difference is 0 at both a and b. Consequently Rolle's theorem applies to g and there is a point c between a and b such that $g'(c) = 0$. .

Further

$g(a) = f(a) - \ell(a) = f(a) - f(a) = 0$ and $g(b) = f(b) - \ell(b) = f(b) - f(b) = 0$.

So Rolle's theorem applies to g. This means there is a point c between a and b such that $g'(c) = 0$. But $g'(c) = f'(c) - \ell'(c) = 0$ which means using (30.1)

$$f'(c) = \ell'(c) = \frac{f(b) - f(a)}{b - a}.$$

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Mostly the MVT gets used to prove other theorems. But we can look at an example or two to see how it works.

EXAMPLE 30.1.5. Show how the MVT applies to $f(x) = x^3 - 6x + 1$ on $[0, 3]$.

Solution.   Check the two conditions (hypotheses)

1. f is continuous on the closed interval $[0, 3]$ because it is a polynomial;

   f is differentiable on the open interval $(0, 3)$ again because it is a polynomial;

So the MVT applies: There is some point c between 0 and 3 so that

$$f'(c) = \frac{f(3) - f(0)}{3 - 0} = \frac{10 - 1}{3} = 3.$$

Now

$$f'(x) = 3x^2 - 6 = 3$$
$$3x^2 = 9$$
$$x = \pm\sqrt{3}$$

Only $x = \sqrt{3}$ is in the interval, so this is the value of c.

EXAMPLE 30.1.6. Show there does not exist a differentiable function on $[1, 5]$ with $f(1) = -3$ and $f(5) = 9$ with $f'(x) \leq 2$ for all x.

Solution.   The MVT would apply to such a function f: So there is some point c between 1 and 5 so that

$$f'(c) = \frac{f(5) - f(1)}{5 - 1} = \frac{9 - (-3)}{4} = 3.$$

But supposedly $f'(x) \leq 2$ for all x. Contradiction. So no such f can exist.

Using the MVT: Increasing and Decreasing Functions

IRST LET'S BE clear on what increasing and decreasing functions are.

YOU TRY IT 30.8. How would you modify the definition to describe f is **decreasing on** I?

As noted, the key value of the MVT is in proving other results.

Answer to YOU TRY IT 30.8 . f is **decreasing on** I if whenever a and b are in I and $a < b$, then $f(a) > f(b)$.

Proof. We'll prove (2). So assume $f'(x) < 0$ for all x in I. Let a and b be any two points in I with $a < b$. To show that f is decreasing, we need to show that $f(b) < f(a)$.
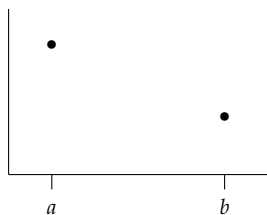
But f is differentiable on I so it is continuous on $[a, b]$ and differentiable on (a, b) so the MVT applies to f on $[a, b]$. So there's a point c in (a, b) so that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

or

$$f(b) - f(a) = f'(c)(b - a) = (-)(+) < 0.$$

But

$$f(b) - f(a) < 0 \Rightarrow f(b) < f(a)$$
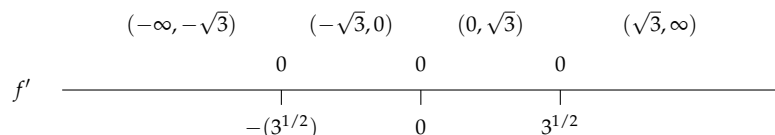
which is what we wanted to show. $\square$

YOU TRY IT 30.9. Prove case (1) of the Increasing/Decreasing Test where $f'(x) > 0$.

EXAMPLE 30.1.9. Let $f(x) = x^4 - 6x^2 + 1$. Where is f increasing? Decreasing? Where does it have relative extrema?
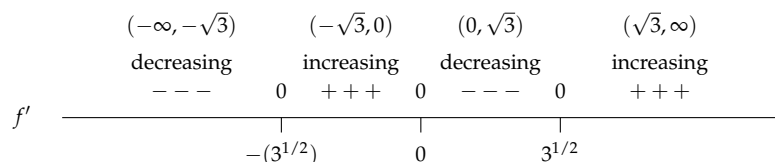
Solution. Use the Increasing/Decreasing Test. Find the derivative and the critical points ($f'(x) = 0$ or DNE).

$$f'(x) = 4x^3 - 12x = 4x(x^2 - 3) = 4x(x - \sqrt{3})(x + \sqrt{3}) = 0 \quad \text{at} \quad x = \pm\sqrt{3}, 0.$$

Now record this information on a number line for easy reference.

$$(-\infty, -\sqrt{3}) \qquad (-\sqrt{3}, 0) \qquad (0, \sqrt{3}) \qquad (\sqrt{3}, \infty)$$

$$0 \qquad\qquad 0 \qquad\qquad 0$$

f' ————————————————————————————————————

$$-(3^{1/2}) \qquad\qquad 0 \qquad\qquad 3^{1/2}$$

Now determine the sign of $f'(x)$ between and beyond the critical points. Here we use the IVT to know that the only places that the derivative can change sign are at the critical points because the derivative is continuous. Just plug in values in the appropriate intervals. $f'(-2) = -8$, $f'(-1) = +8$, $f'(1) = -8$, and $f'(2) = +8$.

$$(-\infty, -\sqrt{3}) \qquad (-\sqrt{3}, 0) \qquad (0, \sqrt{3}) \qquad (\sqrt{3}, \infty)$$

decreasing $\qquad$ increasing $\quad$ decreasing $\qquad$ increasing

$$- - - \qquad 0 \quad + + + \quad 0 \quad - - - \quad 0 \qquad + + +$$

f' ————————————————————————————————————

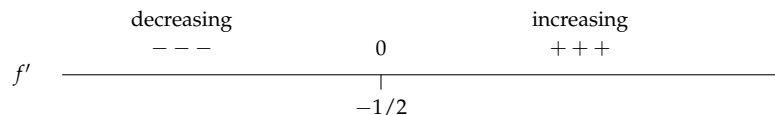$$-(3^{1/2}) \qquad\qquad 0 \qquad\qquad 3^{1/2}$$

Using interval notation: f is increasing on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$ and it is decreasing on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$.

EXAMPLE 30.1.10 (This example is also used in the next section.). Let $f(x) = xe^{2x}$. Where is f increasing? Decreasing?

Solution. Use the Increasing/Decreasing Test. Find the derivative and the critical points.

$$f'(x) = e^{2x} + 2xe^{2x} = e^{2x}[1 + 2x] = 0 \quad \text{at} \quad x = -1/2.$$

Set up the number line and determine the sign of $f'(x)$ on either side of the critical point. $f'(-1) = -e^{-2} < 0$ and $f'(0) = 1$.

decreasing $\qquad\qquad\qquad\qquad\qquad$ increasing

$$- - - \qquad\qquad\qquad 0 \qquad\qquad + + +$$

f' ————————————————————————————————————

$$-1/2$$

Using interval notation: f is increasing on $(-1/2, \infty)$ and it is decreasing on $(-\infty, -1/2)$.