Concavity

KNOWING WHERE A function is increasing and decreasing gives a us a good sense of the shape of its graph. We can refine that sense of shape by determining which way the function bends. This bending is called the **concavity** of the function. I am going to use a different definition of concavity than the text. But the definition in the text will be a consequence (or theorem) resulting from this definition. So they end up the same.

Curves that bend up so that they are 'cupped up' or 'hold water' are called **concave up** while curves that bend down or are 'cupped down' so that they 'spill water' are called **concave down**. Parabolas provide the basic examples.

Figure 32.1: Concave up (left) and concave down curves. Note **where** the tangents are in relation to each curve. Then note how the slope of the tangent lines is changing for each curve.

Notice the location of the tangents to each type of curve.

DEFINITION 32.0.1. If all of the tangents lie below the graph of $y = f(x)$ on an interval I, then f is **concave up** on I.

If all of the tangents lie above the graph of $y = f(x)$ on an interval I, then f is **concave down** on I.

Now look at what has to happen to the **slope** of f for it to be concave up: The slope (i.e., $f'(x)$) must be increasing so its derivative must be positive. That is

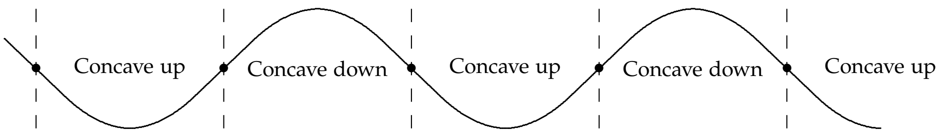
$$\begin{aligned}\text{Concave up} &\Rightarrow \text{slope of } f \text{ is increasing}\\ &\Rightarrow f'(x) \text{ is increasing}\\ &\Rightarrow \frac{d}{dx}(f'(x)) > 0\\ &\Rightarrow f''(x) > 0\end{aligned}$$

$$\begin{aligned}\text{Concave down} &\Rightarrow \text{slope of } f \text{ is decreasing}\\ &\Rightarrow f'(x) \text{ is decreasing}\\ &\Rightarrow \frac{d}{dx}(f'(x)) < 0\\ &\Rightarrow f''(x) < 0\end{aligned}$$

This gives us

THEOREM 32.0.2 (The Concavity Test). Let f be a function whose second derivative exists on an interval I.

1. If $f''(x) > 0$ for all points in I, then f is concave up on I.
2. If $f''(x) < 0$ for all points in I, then f is concave down on I.



Notice in the curve above that there are several places where the concavity switches. The second derivative must be 0 (or else it does not exist) at the point since the curve is not bending either way. We give these points where the concavity changes a special name.

DEFINITION 32.0.3. A point P is called a **point of inflection** for f if f changes concavity at P.

We can find such points by looking for places where the second derivative, $f''(x)$, changes sign.

EXAMPLE 32.0.4. Find the intervals of concavity and the inflection points for $f(x) = x^4 - 6x^2 + 1$. Sketch a graph that includes relative extrema, critical numbers, and inflections.

Solution. Begin with the first derivative. Previously we saw

$$f'(x) = 4x^3 - 12x = 4x(x^2 - 3) = 4x(x - \sqrt{3})(x + \sqrt{3}) = 0 \quad \text{at} \quad x = \pm\sqrt{3}, 0.$$

	dec	l min	inc	l min	dec	l max	inc
f'	$---$	0	$+++$	0	$---$	0	$+++$
		$-(3^{1/2})$		0		$3^{1/2}$	

Now we can take the second derivative.

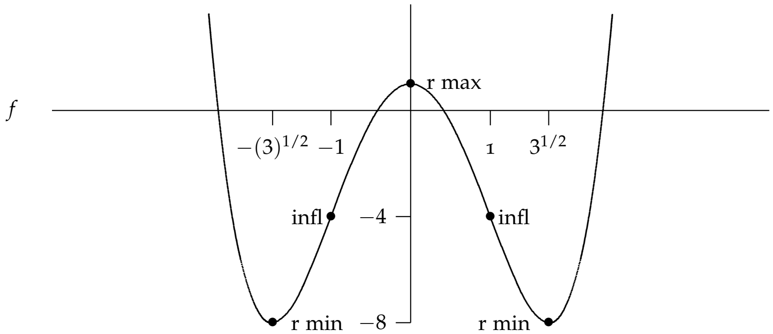
$$f''(x) = 12x^2 - 12 = 12(x^2 - 1) = 0 \quad \text{at} \quad x = \pm 1.$$

These are the potential inflection points; we still have to determine whether the sign of f'' actually changes at these points (so that the concavity changes). Check that now. You should find the behavior summarized below.

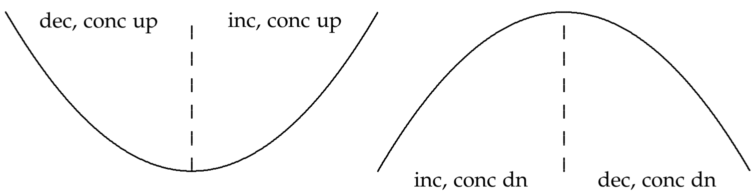
	concave up	inflect	conc dn	inflect	concave up
f''	$++++$	0	$----$	0	$++++$
		-1		1	

To make a graph we will plot just the critical numbers and the inflection points and then connect them appropriately based on the information from the two derivatives. Evaluating the original function f at the critical and inflection points, we find

- $f(-\sqrt{3}) = 9 - 18 + 1 = -8$ and $f(\sqrt{3}) = 9 - 18 + 1 = -8$.
- $f(0) = 1$.
- $f(1) = 1 - 6 + 1 = -4$ and $f(-1) = 1 - 6 + 1 = -4$.



Note The various combinations of increasing/decreasing concave up/down are illustrated below.



The graph indicates that for smooth (differentiable twice) functions, that at a local min the function must be concave up and at a local max the function is concave down. Interpreting concavity by using the second derivative leads to

THEOREM 32.0.5 (Second Derivative Test). Assume that f is a function so that $f'(c) = 0$ and f'' exists on an open interval containing c.

(*a*) If $f''(c) > 0$, then f has a local min at c.

(*b*) If $f''(c) < 0$, then f has a local max at c.

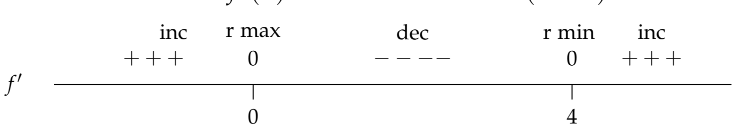
(*c*) If $f''(c) = 0$, then the test fails there may be a local max, or min, or neither at c.

Possibility (3) is why the First Derivative Test is more useful when classifying critical numbers. In this regard, the Second Derivative Test acts mostly as a check to ensure that you have not made an error in classification.

YOU TRY IT 32.1. From Test 3A: Do a complete graph of $f(x) = x^4 + 4x^3 + 10$. Indicate all extrema, inflections, etc.

EXAMPLE 32.0.6. Do a complete graph of $y = f(x) = x^5 - 5x^4$. Indicate all extrema, inflections, etc.

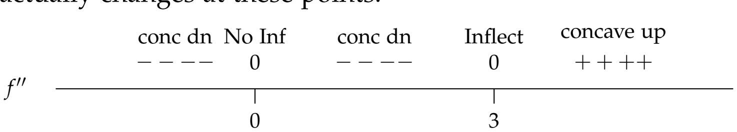
Solution. Critical numbers: $f'(x) = 5x^4 - 20x^3 = 5x^3(x-4) = 0$ at $x = 0,\ 4$.



Now take the second derivative.

$$f''(x) = 20x^3 - 60x^2 = 20x^2(x-3) = 0 \quad \text{at} \quad x = 0,\ 3.$$

These are the potential inflection points; we still have to determine whether the sign of f'' actually changes at these points.



Aside: Apply the second derivative test to the critical numbers. Notice that $f''(4) > 0$, therefore by the second derivative test there is a local min at $x = 4$.

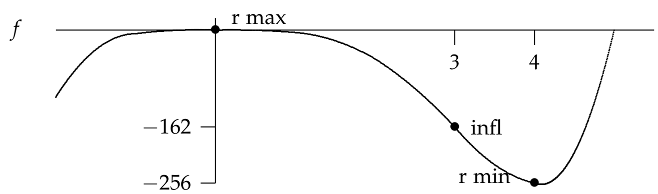
However, $f''(0) = 0$, so the second derivative test fails. Nonetheless, the first derivative test tell us that there is relative max at 0.

To make a graph we will plot just the critical numbers and the inflections and then connect them appropriately based on the information from the two derivatives.

(*a*) $f(0) = 0$.

(*b*) $f(4) = 1024 - 1280 = -256$.

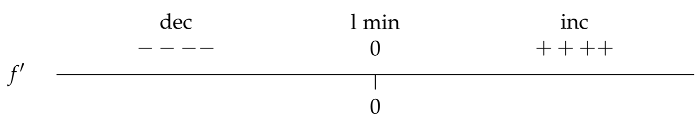
(*c*) $f(3) = 243 - 405 = -162$.



EXAMPLE 32.0.7. Do a complete graph of $y = f(x) = \frac{x^2}{x^2+1}$. Indicate all extrema, inflections, etc.

Solution. First locate critical numbers:

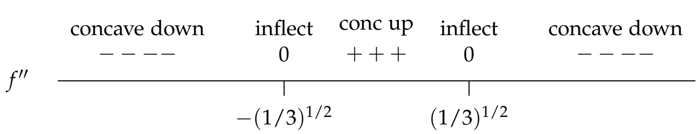
$$f'(x) = \frac{2x(x^2+1) - x^2(2x)}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2} = 0 \quad \text{at } x = 0.$$



Check concavity with the second derivative.

$$f''(x) = \frac{2(x^2+1)^2 - 2x(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{2(x^2+1) - 8x^2}{(x^2+1)^3} = \frac{2-6x^2}{(x^2+1)^3} = 0$$

at $x = \pm 1/\sqrt{3} \approx \pm 0.577$.

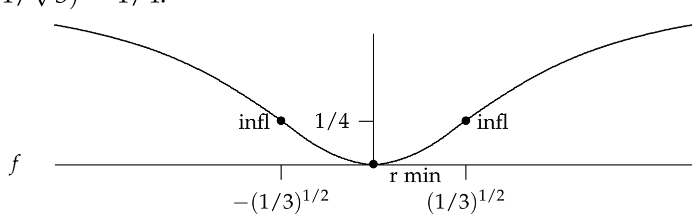


Plot the critical numbers and the inflections and then connect them appropriately based on the information from the two derivatives.

(*a*) $f(0) = 0$.

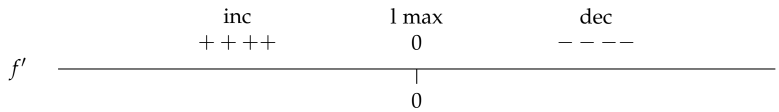
(*b*) $f(-1/\sqrt{3}) = \frac{1/3}{4/3} = 1/4$.

(*c*) $f(1/\sqrt{3}) = 1/4$.



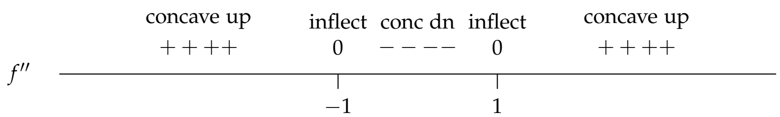
EXAMPLE 32.0.8. Do a complete graph of $y = f(x) = e^{-x^2/2}$. Indicate all extrema, inflections, etc.

Solution. Locate critical numbers: $f'(x) = -xe^{-x^2/2} = 0$ at $x = 0$.



Check concavity with the second derivative.

$$f''(x) = -e^{-x^2/2} - xe^{-x^2/2}(-x) = (x^2-1)e^{-x^2/2} = 0 \quad \text{at} \quad x = \pm 1.$$

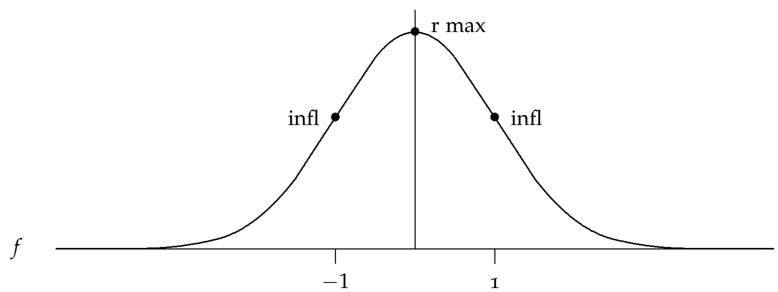


Plot the critical numbers and the inflections and then connect them appropriately based on the information from the two derivatives.

(*a*) $f(0) = e^0 = 1$.

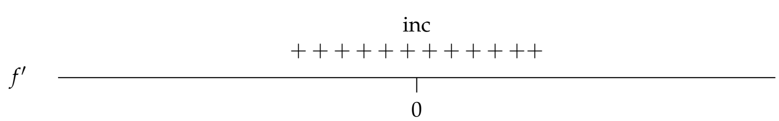
(*b*) $f(-1) = e^{-1/2} \approx 0.607$.

(*c*) $f(1) = e^{-1/2} \approx 0.607$.

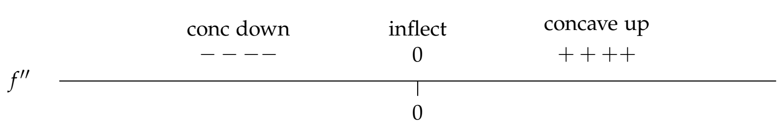


EXAMPLE 32.0.9. Here's a quick one with an interesting point: Do a complete graph of $y = f(x) = x^3 + x$.

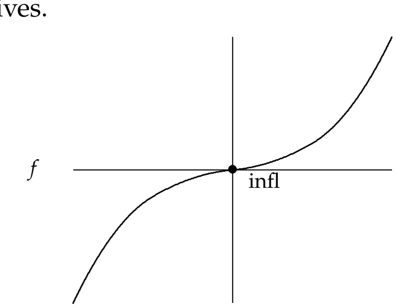
Solution. Locate critical numbers: $f'(x) = 3x^2 + 1 \neq 0$; there are no critical numbers.



Check concavity with the second derivative: $f''(x) = 6x = 0$ at $x = 0$.



The only point to plot is $f(0) = 0$. The shape of the curve is determined by the first and second derivatives.



YOU TRY IT 32.2. Graph each of the following:

(*a*) $f(x) = \frac{1}{x^2+1}$. Hint: The graph will look very similar to the exponential graph we did above.

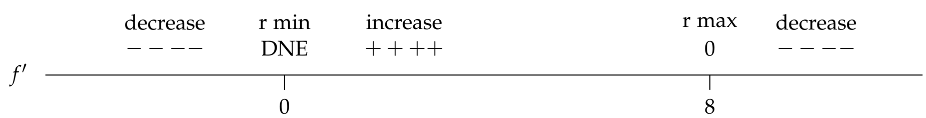
(*b*) $f(x) = x^4 - 4x^3$. Hint: It has critical numbers at $x = 0$ and 3 and inflections at $x = 0$ and 2.

(*c*) $f(x) = x - \cos x$. It has critical numbers but no extrema. It has lots of inflections.

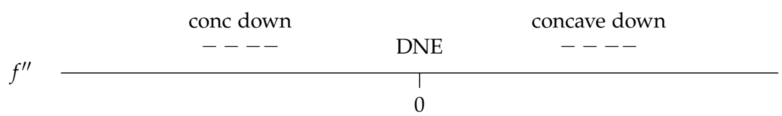
(*d*) Challenge: Graph $f(x) = (x^3 - 8)^{1/3}$.

EXAMPLE 32.0.10. Here's another quick one with a twist: Do a complete graph of $y = f(x) = 3x^{2/3} - x$.

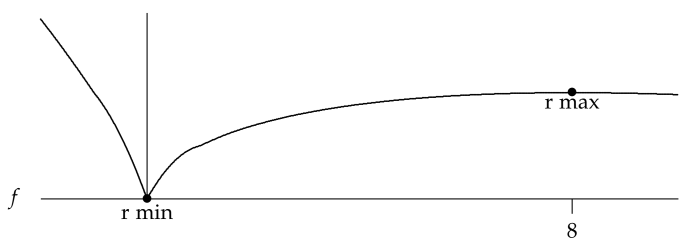
Solution. Locate critical numbers: $f'(x) = 2x^{-1/3} - 1 = \frac{2}{x^{1/3}} - 1 = 0$, so $\frac{2}{x^{1/3}} = 1$ so $\frac{1}{x^{1/3}} = \frac{1}{2}$ so $x^{1/3} = 2$ or $x = 8$. Also $f'(x)$ DNE at $x = 0$.



Check concavity with the second derivative: $f''(x) = \frac{2/3}{x^{4/3}} \neq 0$, but $f''(x)$ DNE at $x = 0$.



The only points to plot is $f(0) = 0$ and $f(8) = 4$. The shape of the curve is determined by the first and second derivatives.



EXAMPLE 32.0.11. Below I give you information about the first and second derivatives of two *continuous* functions. For each, sketch a function that would have dervatives like those given. Indicate on your graph which points are local extrema and which are inflections. $*$ indicates that the derivative does not exist at the point (though the original function does). You will need to make up values for the critical and inflection points consistent with the information supplied.