Graphing Rational Functions

Let's use all of the material we have developed to graph some rational functions.

EXAMPLE 34.0.1. Graph $y = f(x) = \frac{x^2+3x-3}{x^2}$. Include both vertical and horizontal asymptotes.

Solution. First determine the domain: $f(x)$ is rational and is not defined where the denominator is 0. That's at $x = 0$. This leads us to look VAs, RDs, and HAs
VA: Since the function is not defined at $x = 0$, we look to see if there is a VA there.

$\lim_{x \to 0^+} \frac{\overbrace{x^2+3x-3}^{-3}}{\underbrace{x^2}_{0^+}} = -\infty$ and $\lim_{x \to 0^-} \frac{\overbrace{x^2+3x-3}^{-3}}{\underbrace{x^2}_{0^+}} = -\infty$; so VA at $x = 0$. You may wish to indicate the VA in your graph at this point.

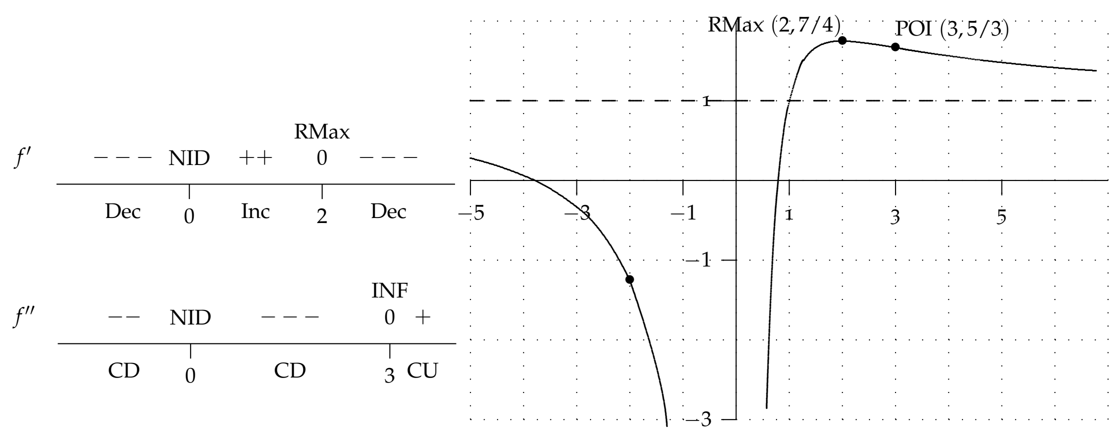
HA and End Behavior: Using dominant powers, $\lim_{x \to +\infty} \frac{x^2+3x-3}{x^2} \overset{\text{HP}}{=} \lim_{x \to +\infty} \frac{x^2}{x^2} = 1$ and $\lim_{x \to -\infty} \frac{x^2+3x-3}{x^2} \overset{\text{HP}}{=} \lim_{x \to -\infty} \frac{x^2}{x^2} = 1$. So HA at $y = 1$. You may wish to indicate the HA in your graph at this point.

Critical points, local extrema, increasing/decreasing behavior.
$f'(x) = \frac{(2x+3)x^2 - (x^2+3x-3)2x}{x^4} = \frac{(2x^2+3x) - (2x^2+6x-6)}{x^3} = \frac{-3x+6}{x^3} =$ 0 at $x = 2$ (and $x = 0$ NID).

Inflections and concavity.
$f''(x) = \frac{-3x^3 - (-3x+6)3x^2}{x^6} = \frac{-3x - (-3x+6)(3)}{x^4} = \frac{6x-18}{x^4} = 0$ at $x = 3$ (and $x = 0$ NID).

Evaluate f at key points. $f(2) = \frac{4+6-3}{4} = \frac{7}{4}$, $f(3) = \frac{9+9-3}{9} = \frac{5}{3}$.
Notice that the inflection is almost imperceptible in the graph.

We will need another point to graph when $x < 0$. $f(-2) = \frac{4-6-3}{4} = -1.25$.



EXAMPLE 34.0.2. Graph $y = f(x) = \frac{x^2}{x-4}$. Include both vertical and horizontal asymptotes, if they exist.

Solution. Notice that $x = 4$ is not in the domain. Check there for a VA.

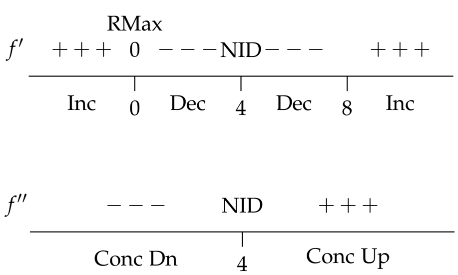
VA: $\lim\limits_{x\to 4+} \dfrac{\overbrace{x^2}^{16}}{\underbrace{x-4}_{0^+}} = +\infty$ and $\lim\limits_{x\to 4-} \dfrac{\overbrace{x^2}^{16}}{\underbrace{x-4}_{0^-}} = -\infty$; so VA at $x = 4$.

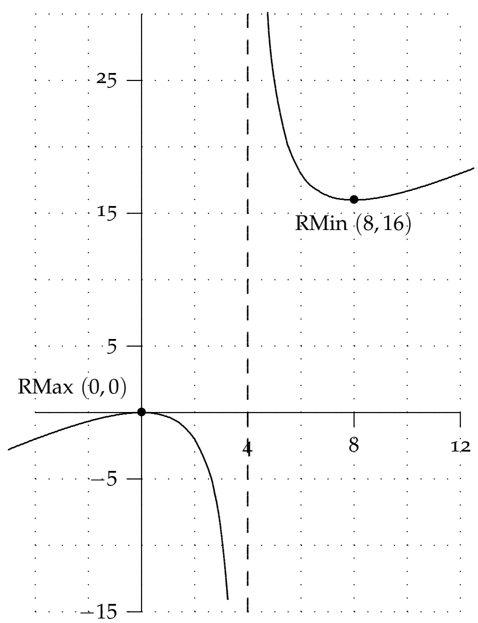
HA and End Behavior: Since the function is rational, using dominant powers, $\lim\limits_{x\to+\infty} \dfrac{x^2}{x-4} \overset{\text{HP}}{=} \lim\limits_{x\to+\infty} \dfrac{x^2}{x} = \lim\limits_{x\to+\infty} x = +\infty$ and $\lim\limits_{x\to-\infty} \dfrac{x^2}{x-4} \overset{\text{HP}}{=} \lim\limits_{x\to-\infty} \dfrac{x^2}{x} = \lim\limits_{x\to-\infty} x = -\infty$. So there are no HAs, but we do know what is happening at either end.

Critical points, local extrema, increasing/decreasing behavior.
$f'(x) = \dfrac{2x(x-4)-x^2}{(x-4)^2} = \dfrac{x^2-8x}{(x-4)^2} = \dfrac{x(x-8)}{(x-4)^2} = 0$ at $x = 0, 8$ (and $x = 4$ NID).

Inflections and concavity.
$f''(x) = \dfrac{(2x-8)(x-4)^2-(x^2-8x)(2)(x-4)}{(x-4)^4} = \dfrac{(2x-8)(x-4)-(x^2-8x)(2)}{(x-4)^3} = \dfrac{(2x^2-16x+32-2x^2+16x)}{(x-4)^3} = \dfrac{32}{(x-4)^3} \neq 0$ (but $x = 4$ NID). So there are no inflections. But the concavity may still switch on either side of $x = 4$.

Evaluate f at key points. $f(0) = 0$ and $f(8) = \frac{64}{8-4} = 16$.





EXAMPLE 34.0.3. Graph $y = f(x) = \frac{2x}{x+2}$. Include both vertical and horizontal asymptotes.

Solution. This time $x = -2$ is not in the domain.

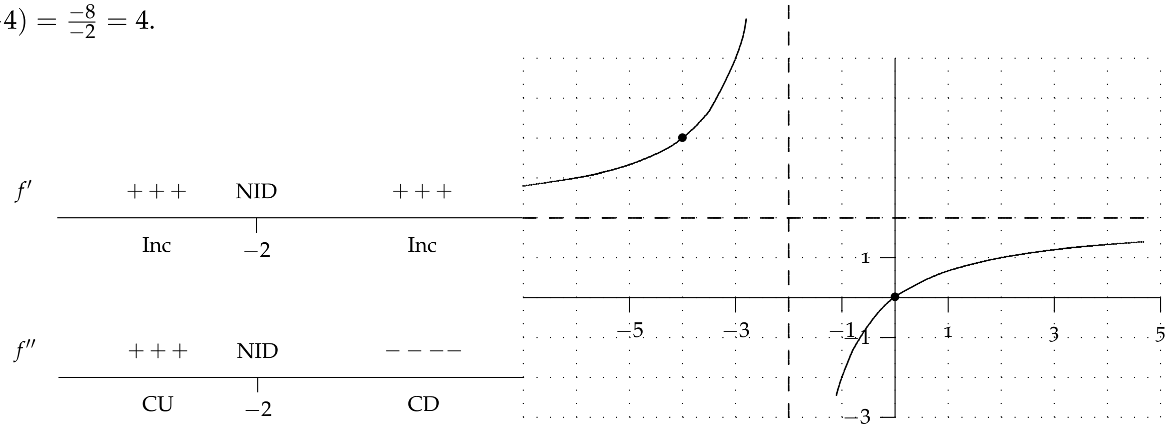
VA: $\lim\limits_{x\to-2^+} \dfrac{\overbrace{2x}^{-4}}{\underbrace{x+2}_{0^+}} = -\infty$ and $\lim\limits_{x\to-2^+} \dfrac{\overbrace{2x}^{-4}}{\underbrace{x+2}_{0^-}} = +\infty$; so VA at $x = -2$.

HA: Using dominant powers, $\lim\limits_{x\to+\infty} \dfrac{2x}{x+2} \overset{\text{HP}}{=} \lim\limits_{x\to+\infty} \dfrac{2x}{x} = 2$ and $\lim\limits_{x\to-\infty} \dfrac{2x}{x+2} \overset{\text{HP}}{=} \lim\limits_{x\to-\infty} \dfrac{2x}{x} = 2$. So HA at $y = 2$.

Critical points, local extrema, increasing/decreasing behavior.
$f'(x) = \dfrac{2(x+2)-2x}{(x+2)^2} = \dfrac{4}{(x+2)^2} \neq 0$ (and $x = -2$ NID).

Inflections and concavity.
$f''(x) = \dfrac{-8}{(x+2)^3} \neq 0$ (and $x = -2$ NID).

Evaluate f at key points. There are none! Choose on each side of VA. $f(0) = 0$ and $f(-4) = \frac{-8}{-2} = 4$.



EXAMPLE 34.0.4. Graph $y = f(x) = \frac{2x}{x^2+1}$. Include both vertical and horizontal asymptotes.

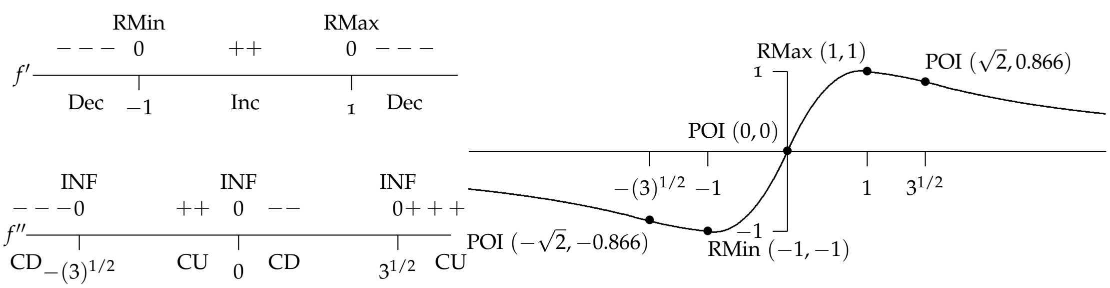
Solution. VA: None, the denominator of this rational function is never 0.

HA: Using dominant powers, $\lim\limits_{x\to+\infty} \dfrac{2x}{x^2+1} \stackrel{\text{HP}}{=} \lim\limits_{x\to+\infty} \dfrac{2x}{x^2} = \lim\limits_{x\to+\infty} \dfrac{2}{x} = 0$ and $\lim\limits_{x\to-\infty} \dfrac{2x}{x^2+1} \stackrel{\text{HP}}{=} \lim\limits_{x\to-\infty} \dfrac{2x}{x^2} = \lim\limits_{x\to-\infty} \dfrac{2}{x} = 0$. So HA at $y = 0$.

Critical points, local extrema, increasing/decreasing behavior.
$f'(x) = \dfrac{2(x^2+1)-(2x)2x}{(x^2+1)^2} = \dfrac{2-2x^2}{(x^2+1)^2} = 0$ at $x = \pm 1$.

Inflections and concavity.
$f''(x) = \dfrac{-4x(x^2+1)^2 - (2-2x^2)2(x^2+1)2x}{(x^2+1)^4} = \dfrac{4x^3-12x}{(x^2+1)^3} = \dfrac{4x(x^2-3)}{(x^2+1)^3} = 0$ at $x = 0, \pm\sqrt{3}$.

Evaluate f at key points. $f(1) = 1$, $f(-1) = -1$, $f(0) = 0$, $f(\sqrt{3}) = \frac{\sqrt{3}}{2} \approx 0.866$ and $f(-\sqrt{3}) = -\frac{\sqrt{3}}{2} \approx -0.866$.



EXAMPLE 34.0.5. Graph $y = f(x) = \frac{2x^2}{(x-1)^2}$, where $x \neq 1$. Include both vertical and horizontal asymptotes.

Solution. The function is not defined at $x = 1$.

VA: Look near $x = 1$. $\lim\limits_{x\to 1^+} \dfrac{\overbrace{2x^2}^{1}}{\underbrace{(x-1)^2}_{0^+}} = +\infty$ and $\lim\limits_{x\to 1^-} \dfrac{\overbrace{2x^2}^{1}}{\underbrace{(x-1)^2}_{0^+}} = +\infty$.

HA: Using dominant powers,

$$\lim_{x\to+\infty} \frac{2x^2}{(x-1)^2} = \lim_{x\to+\infty} \frac{2x^2}{x^2-2x+1} \stackrel{\text{HP}}{=} \lim_{x\to+\infty} \frac{2x^2}{x^2} = 2$$

and

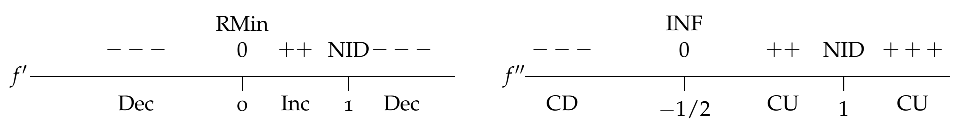
$$\lim_{x\to-\infty} \frac{2x^2}{(x-1)^2} \stackrel{\text{HP}}{=} \lim_{x\to-\infty} \frac{2x^2}{x^2} = 2.$$

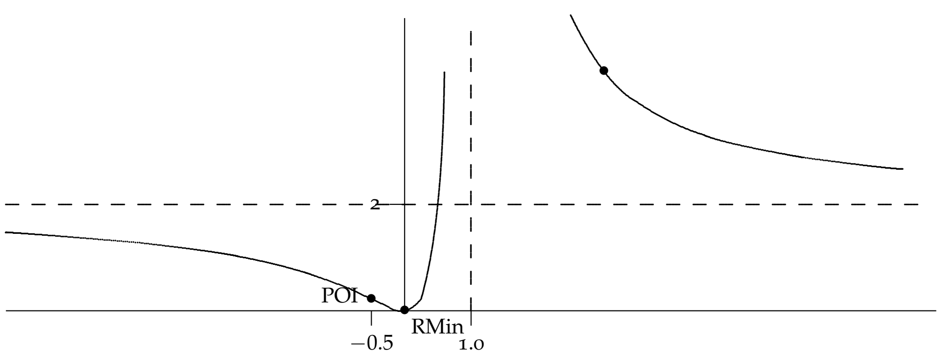
So HA at $y = 2$.

Critical points, local extrema, increasing/decreasing behavior.
$f'(x) = \dfrac{4x(x-1)^2 - (2x^2)2(x-1)}{(x-1)^4} = \dfrac{4x^2-4x-4x^2}{(x-1)^3} = \dfrac{-4x}{(x-1)^3} = 0$ at $x = 0$, $x = 1$ NID.

Inflections and concavity.
$f''(x) = \dfrac{-4(x-1)^3 - 4x(3)2(x-1)^2}{(x-1)^6} = \dfrac{-4x+4+12x}{(x-1)^4} = \dfrac{8x+4}{(x-1)^4} = 0$ at $x = -1/2$, $x = 1$ NID.

Evaluate f at key points. $f(0) = 0$, $f(-1/2) = 2/9$ and we need a point when $x > 1$ on the other side of the VA: $f(3) = 18/4 = 4.5$.





EXAMPLE 34.0.6. Graph $y = f(x) = \frac{x}{x^2-4}$, where $x \neq \pm 2$. Include both vertical and horizontal asymptotes.

Solution. The function is not defined at $x = \pm 2$.

VA: Look near $x = 2$. $\lim\limits_{x\to 2^+} \dfrac{\overbrace{x}^{2}}{\underbrace{x^2-4}_{0^+}} = +\infty$ and $\lim\limits_{x\to 2^+} \dfrac{\overbrace{x}^{2}}{\underbrace{x^2-4}_{0^-}} = -\infty$. Now look near $x = -2$. $\lim\limits_{x\to -2^+} \dfrac{\overbrace{x}^{-2}}{\underbrace{x^2-4}_{0^-}} = +\infty$ and $\lim\limits_{x\to -2^-} \dfrac{\overbrace{x}^{-2}}{\underbrace{x^2-4}_{0^+}} = -\infty$ VAs: $x = \pm 2$

HA: Using dominant powers,

$$\lim_{x\to+\infty}\frac{x}{x^2-4}\overset{\text{HP}}{=}\lim_{x\to+\infty}\frac{x}{x^2}=\lim_{x\to+\infty}\frac{1}{x}=0$$

and

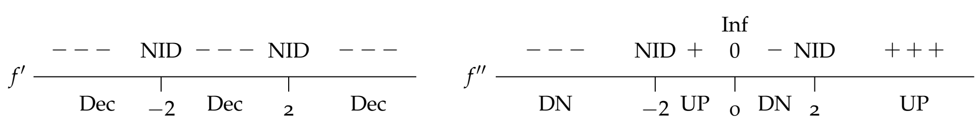
$$\lim_{x\to-\infty}\frac{x}{x^2-4}\overset{\text{HP}}{=}\lim_{x\to-\infty}\frac{1}{x}=0$$

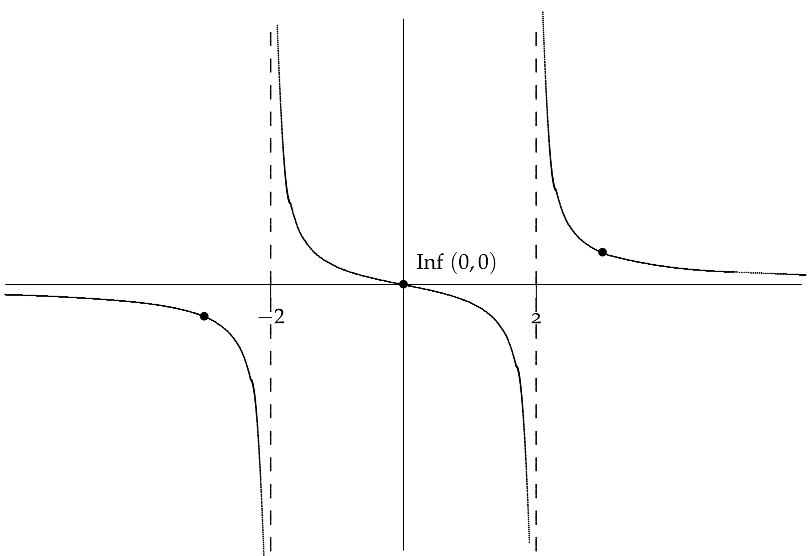
So HA at $y=0$.

Critical points, local extrema, increasing/decreasing behavior.
$f'(x)=\dfrac{x^2-4-2x^2}{(x^2-4)^2}=\dfrac{-x^2-4}{(x^2-4)^2}\neq 0$, $x=\pm 2$ NID.

Inflections and concavity.
$f''(x)=\dfrac{-2x(x^2-4)^2+(x^2+4)2(x^2-4)(2x)}{(x^2-4)^4}=\dfrac{-2x^3+8x+4x^3+16x}{(x^2-4)^3}=\dfrac{2x^3+24x}{(x^2-4)^3}=\dfrac{2x(x^2+12)}{(x^2-4)^3}=0$ at $x=0$, $x=\pm 2$ NID.

Evaluate f at key points. $f(0)=0$, and we need a points when $x>2$ and $x<-2$ on the far side of the VAs :$f(3)=3/5$ and $f(-3)=-3/5$.





YOU TRY IT 34.1. Here is information about the first and second derivatives of a function and its vertical and horizontal asymptotes. Sketch a function that satisfies these conditions. Indicate on your graph which points are local extrema and which are inflections. **NID** means the point is "not in the domain" of the original function. Let $f(0)=-1$ and $\lim_{x\to1^+}f(x)=+\infty$, $\lim_{x\to1^-}f(x)=-\infty$, $\lim_{x\to+\infty}f(x)=1$, and $\lim_{x\to-\infty}f(x)=+\infty$.