Evaluating Limits Using l'Hôpital's Rule

Some useful limits. Before we look at any further examples and techniques for computing limits, here are some very handy limits that you should know. All of these limits come from looking at the graphs of the particular log or exponential function.

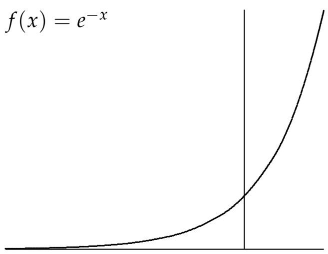


Figure 37.1: The graph of $f(x) = e^x$. $\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to\infty} e^x = +\infty$. So $y = 0$ is HA for $y = e^x$.

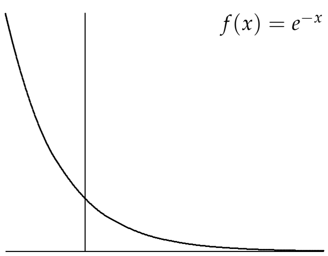


Figure 37.2: The graph of $f(x) = e^{-x}$. $\lim_{x\to\infty} e^{-x} = 0$ and $\lim_{x\to-\infty} e^{-x} = +\infty$. $y = 0$ is HA for $y = e^{-x}$.

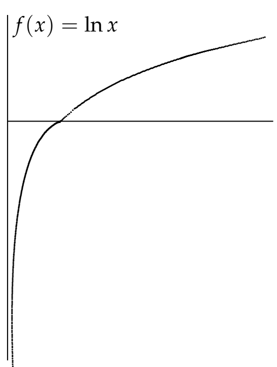


Figure 37.3: The graph of $f(x) = \ln x$. $\lim_{x\to\infty} \ln x = +\infty$ and $\lim_{x\to 0^+} \ln x = -\infty$ and $x = 0$ is VA for $y = \ln x$.

THEOREM 37.0.1 (The End Behavior of the Natural Log and Exponential Functions)**.** The end behavior of e^x and e^{-x} on $9 - \infty, \infty)$ and $\ln x$ on $(0, +\infty)$ is given by

$$\lim_{x\to\infty} e^x = +\infty \quad \text{and} \quad \lim_{x\to-\infty} e^x = 0$$

$$\lim_{x\to\infty} e^{-x} = 0 \quad \text{and} \quad \lim_{x\to-\infty} e^{-x} = +\infty$$

$$\lim_{x\to 0^+} \ln x = -\infty \quad \text{and} \quad \lim_{x\to\infty} \ln x = +\infty$$

Introduction: Indeterminate Forms

Most of the interesting limits in Calculus I have the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Remember that we say that such limits have **indeterminate form**.

Such limits require "more work" to evaluate them. This work might be factoring, using conjugates, using known limits, or dividing by the highest power of x. Here are three common types of indeterminate limits:

1. $\dfrac{0}{0}$ form: If we let $x = 2$ in
$$\lim_{x\to 2}\frac{x^2-4}{x-2},$$
we obtain a meaningless $\dfrac{0}{0}$ expression.

2. $\dfrac{\infty}{\infty}$ form: If we let $x \to \infty$ in
$$\lim_{x\to\infty}\frac{2x^2-4}{3x^2+9},$$
we obtain a meaningless $\dfrac{\infty}{\infty}$ expression.

3. and a new type of indeterminate form $\infty \cdot 0$: If we let $x = \infty$ in
$$\lim_{x\to\infty} xe^{-x},$$
we end up with a meaningless $\infty \cdot 0$ expression. Each of these limits requires

'more work' to evaluate them, such as factoring or focusing on highest powers. Now we describe a simple method called l'Hôpital's Rule to evaluate limits, at least limits of the first two types.

See Chapter 4.7 in *Briggs & Stratton*.

The Indeterminate Form $\frac{0}{0}$.

THEOREM 37.0.2 (l'Hôpital's Rule). Let f and g be differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$
provided the limit on the right side exists or is $\pm\infty$. This also applies to **one-sided limits** and to limits as $x \to \infty$ or $x \to -\infty$

EXAMPLE 37.0.3. We could evaluate the following indeterminate limit by factoring:
$$\lim_{x\to 2}\frac{x^2-4^{\nearrow 0}}{x-2_{\searrow 0}} = \lim_{x\to 2}\frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2} x+2 = 4.$$
But we could also use l'Hôpital's rule:
$$\lim_{x\to 2}\frac{x^2-4}{x-2} \overset{\text{l'Ho}}{=} \lim_{x\to 2}\frac{2x}{1} = 4$$
which is pretty easy. Just remember to take the derivatives of the numerator and denominator separately.

EXAMPLE 37.0.4. This technique can be applied to problems where our old techniques failed. Here are a few more.

1. $\lim\limits_{x\to 1} \dfrac{1-x^{\nearrow 0}}{\ln x_{\searrow 0}} \stackrel{\text{l'Ho}}{=} \lim\limits_{x\to 1} \dfrac{-1}{\frac{1}{x}} = \lim\limits_{x\to 1} -x = -1$

2. $\lim\limits_{x\to 1} \dfrac{4^x-2^x-2^{\nearrow 0}}{x-1_{\searrow 0}} \stackrel{\text{l'Ho}}{=} \lim\limits_{x\to 1} \dfrac{4^x\ln 4-2^x\ln 2}{1} = 4\ln 4 - 2\ln 2$

3. $\lim\limits_{x\to 0} \dfrac{1-\cos 3x^{\nearrow 0}}{2x^2_{\searrow 0}} \stackrel{\text{l'Ho}}{=} \lim\limits_{x\to 0} \dfrac{3\sin 3x^{\nearrow 0}}{4x_{\searrow 0}} = \lim\limits_{x\to 0} \dfrac{9\cos 3x}{4} = \dfrac{9}{4}$

4. $\lim\limits_{x\to 0} \dfrac{x^2+x^{\nearrow 0}}{e^x-1_{\searrow 0}} \stackrel{\text{l'Ho}}{=} \lim\limits_{x\to 0} \dfrac{2x+1}{e^x} = \dfrac{2}{1} = 2$

5. $\lim\limits_{x\to 0} \dfrac{x^2+x^{\nearrow 0}}{e^x_{\searrow 1}} = 0$. Here l'Hôpital's rule does not apply. The limit can be evaluated since the denominator is not approaching 0.

6. $\lim\limits_{x\to 3^+} \dfrac{x-3^{\nearrow 0}}{\ln(2x-5)_{\searrow 0}} \stackrel{\text{l'Ho}}{=} \lim\limits_{x\to 3^+} \dfrac{1}{\frac{2}{2x-5}} = \lim\limits_{x\to 3^+} \dfrac{2x-5}{2} = \dfrac{1}{2}$.

7. $\lim\limits_{x\to 5} \dfrac{\sqrt{10+3x}-5^{\nearrow 0}}{x-5_{\searrow 0}} \stackrel{\text{l'Ho}}{=} \lim\limits_{x\to 5} \dfrac{\frac{3}{2\sqrt{10+3x}}}{1} = \dfrac{3}{10}$.

Why Should l'Hôpital's Rule Be True?

Here's a proof of a simpler version of l'Hôpital's rule. It makes use of the definition of the derivative.

THEOREM 37.0.5 (l'Hôpital's Rule—Simple Version). Let f and g be differentiable on an open interval I containing a with $g'(a) \neq 0$. Assume that $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} f(x)$ both equal 0. Then
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Proof. Since f and g are differentiable at a, then each is continuous at $x = a$. Therefore, by definition of continuity, $f(a) = \lim\limits_{x\to a} f(x) = 0$ and $g(a) = \lim\limits_{x\to a} g(x) = 0$.

Using the definition of the derivative and that fact that $f(a) = g(a) = 0$, we get
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{\frac{f(x)}{x-a}}{\frac{g(x)}{x-a}} = \lim_{x\to a} \frac{\frac{f(x)-0}{x-a}}{\frac{g(x)-0}{x-a}} = \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \frac{f'(a)}{g'(a)}.$$

That was easy! $\square$

The Indeterminate Form $\frac{\infty}{\infty}$.

l'Hôpital's Rule also applies to indeterminate limits of the form $\frac{\infty}{\infty}$. Specifically

THEOREM 37.0.6 (l'Hôpital's Rule ∞/∞). Let f and g be differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$. If $\lim\limits_{x\to a} f(x) = \pm\infty$ and $\lim\limits_{x\to a} g(x) = \pm\infty$, then
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$
provided the limit on the right side exists or is $\pm\infty$. This also applies to **one-sided limits** and to limits as $x \to \infty$ or $x \to -\infty$

EXAMPLE 37.0.7. We could evaluate the following indeterminate limit by using highest powers:

$$\lim_{x\to+\infty} \frac{2x^2+4^{\nearrow+\infty}}{3x^2+x_{\searrow+\infty}} = \lim_{x\to+\infty} \frac{2x^2}{3x^2} = \frac{2}{3}.$$

But we could also use l'Hôpital's rule:

$$\lim_{x\to+\infty} \frac{2x^2+4^{\nearrow+\infty}}{3x^2+x_{\searrow+\infty}} \overset{\text{l'Ho}}{=} \lim_{x\to+\infty} \frac{4x^{\nearrow+\infty}}{6x+1_{\searrow+\infty}} \overset{\text{l'Ho}}{=} \lim_{x\to+\infty} \frac{4}{6} = \frac{2}{3}.$$

A more interesting example that we could not have done earlier would be

$$\lim_{x\to0^+} \frac{\ln x^{\nearrow-\infty}}{\frac{1}{x}_{\searrow+\infty}} \overset{\text{l'Ho}}{=} \lim_{x\to0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to0^+} \frac{x}{-1} = 0.$$

Repeated use of l'Hôpital's Rule is often required. Make sure to check that it applies at each step.

$$\lim_{x\to\infty} \frac{\overbrace{2e^x+x}^{\infty}}{\underbrace{x^2+7x+1}_{\infty}} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{\overbrace{2e^x+1}^{\infty}}{\underbrace{2x+7}_{\infty}} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{\overbrace{2e^x}^{+\infty}}{\underbrace{2}_{2}} = +\infty.$$

YOU TRY IT 37.1. Try these now.

(a) $\lim_{x\to\infty} \frac{3x^2+7x}{5x^2+11}$ (b) $\lim_{x\to\infty} \frac{-x^2}{e^x}$ (c) $\lim_{x\to\infty} \frac{\ln x}{e^x}$ (d) $\lim_{x\to\infty} \frac{\ln x}{x}$

Answers to YOU TRY IT 37.1 :

(a) $\frac{3}{5}$ (b) 0 (c) 0 (d) 0

The Indeterminate Form: $0\cdot\infty$.

l'Hôpital's rule cannot be directly applied to limits of the form $0\cdot\infty$. However, if we are clever, we can manipulate and rewrite the limit as $0/0$ or ∞/∞ form.

EXAMPLE 37.0.8. Determine $\lim_{x\to0^+} x\ln x^3$.

SOLUTION. Notice that the limit has the indeterminate form $0\cdot\infty$. Rewriting it we can apply l'Hôpital's rule. We want to get it into either of the standard indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$. We do this by changing multiplication into division by the reciprocal. For example, multiplying $\ln x$ by x is the same as dividing $\ln x$ by $\frac{1}{x}$. In other words,

$$\lim_{x\to0^+} x\ln x^3 = \lim_{x\to0^+} 3x\ln x = \lim_{x\to0^+} \frac{3\ln x^{\nearrow-\infty}}{\frac{1}{x}_{\searrow+\infty}} \overset{\text{l'Ho}}{=} \lim_{x\to0^+} \frac{\frac{3}{x}}{-\frac{1}{x^2}} = \lim_{x\to0^+} \frac{3x}{-1} = 0.$$

EXAMPLE 37.0.9. This time determine $\lim_{x\to\infty} x\sin(\frac{1}{x})$.

SOLUTION. Notice that the limit has the indeterminate form '$\infty\cdot0$.' Here we can rewrite the limit as

$$\lim_{x\to\infty} x\sin(\tfrac{1}{x}) = \lim_{x\to\infty} \frac{\sin(\frac{1}{x})^{\nearrow0}}{\frac{1}{x}_{\searrow0}} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{-\frac{1}{x^2}\cos(\frac{1}{x})}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\cos(\frac{1}{x})}{1} = \cos 0 = 1.$$

YOU TRY IT 37.2. Try these. First check whether the limit has the indeterminate form '$\infty\cdot0$'. If so, determine which term makes sense to put in the denominator so that l'Hôpital's rule can be applied. Then solve.

(a) $\lim_{x\to\infty} x^2e^{-x}$

(b) $\lim_{x\to\infty} x\tan(\frac{1}{x})$

(c) $\lim_{x\to0^+} x^2\ln x$

Answers to YOU TRY IT 37.2 :

(a) 0 (b) 1 (c) 0

Using l'Hôpital's rule in Graphing

We can use l'Hôpital's rule when graphing rational functions though using "highest powers" is often easier However, there are some functions involving exponentials and logs that require l'Hôpital's rule to understand their behavior.

EXAMPLE 37.0.10. Graph $y = f(x) = xe^{x/2}$. Include any horizontal asymptotes.

SOLUTION. Horizontal asymptotes (HA) and End Behavior: Use l'Hôpital's rule:

$$\lim_{x\to+\infty} \overbrace{xe^{x/2}}^{\infty\cdot\infty} = +\infty.$$

So there is no HA as $x \to +\infty$. What about as $x \to -\infty$? This limit is harder. Notice that we have an indeterminate form

$$\lim_{x\to-\infty} \overbrace{xe^{x/2}}^{-\infty\cdot 0}.$$

Remember that to deal with such limits we use reciprocals. Here we take the reciprocal of $e^{x/2}$ which will be $\frac{1}{e^{x/2}} = e^{-x/2}$, so

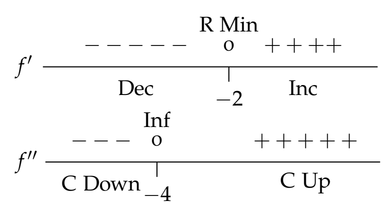
$$\lim_{x\to-\infty} xe^{x/2} = \lim_{x\to-\infty} \frac{x}{\frac{1}{e^{x/2}}} = \lim_{x\to-\infty} \frac{\overbrace{x}^{-\infty}}{\underbrace{e^{-x/2}}_{\infty}} \overset{\text{l'Ho}}{=} \lim_{x\to-\infty} \frac{1}{\underbrace{-\frac{1}{2}e^{-x/2}}_{-\infty}} = 0.$$

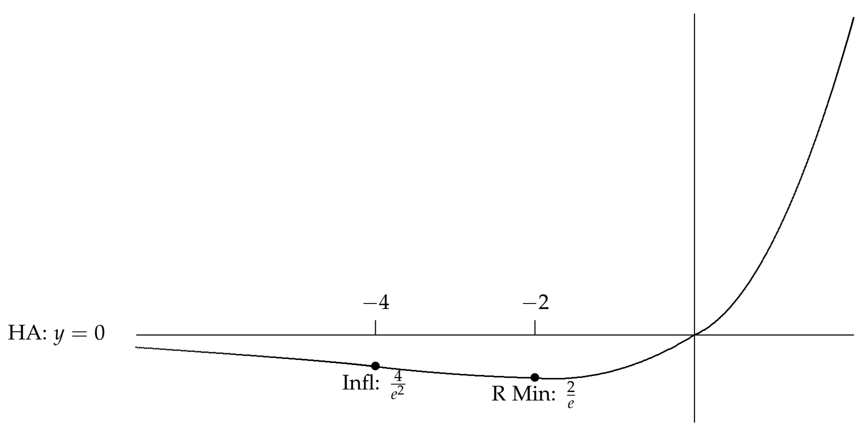
So there is a HA at $y = 0$ as $x \to -\infty$. Next, use the first and second derivatives to get information about the shape of the graph.

$f'(x) = e^{x/2} + \frac{1}{2}xe^{x/2} = (\frac{1}{2}x + 1)e^{x/2} = 0$ at $x = -2$.

$f''(x) = \frac{1}{2}e^{x/2} + (\frac{1}{2}x + 1)(\frac{1}{2}e^{x/2}) = (\frac{1}{4}x + 1)e^{x/2} = 0$ at $x = -4$.

Evaluate f at key points. $f(-2) = -2e^{-1} = -\frac{2}{e} \approx -0.7358$ and $f(-4) = -\frac{4}{e^2} \approx -0.5413$.





EXAMPLE 37.0.11. Graph $y = f(x) = \frac{2x+e^x}{e^x}$. Include both vertical and horizontal asymptotes.

SOLUTION. Horizontal asymptotes (HA) and End Behavior: Use l'Hôpital's rule:

$$\lim_{x\to+\infty} \frac{2x + e^x}{e^x} \overset{\text{l'Ho}}{=} \lim_{x\to+\infty} \frac{2 + e^x}{e^x} \overset{\text{l'Ho}}{=} \lim_{x\to+\infty} \frac{e^x}{e^x} = 1$$

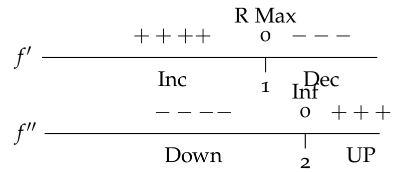
So HA at $y = 1$. Also as $x \to -\infty$ notice that we do not have an indeterminate form: Rather

$$\lim_{x\to-\infty} \frac{(2x + e^x)^{\nearrow -\infty}}{e^x{}_{\searrow 0^+}} = -\infty$$

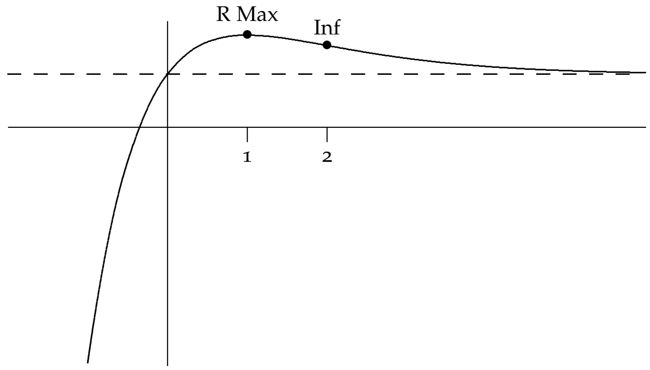
Use the first and second derivatives to get information about the shape of the graph.

$$f'(x) = \frac{(2+e^x)e^x - (2x+x)e^x}{(e^x)^2} = \frac{(2+e^x)-(2x+e^x)}{e^x} = \frac{2-2x}{e^x} = 0 \text{ at } x = 1.$$

$$f''(x) = \frac{-2e^x - (2-2x)e^x}{(e^x)^2} = \frac{-2-(2-2x)}{e^x} = \frac{-4+2x}{e^x} = 0 \text{ at } x = 2.$$



Evaluate f at key points. $f(1) = \frac{2+e}{e} \approx 1.736$ and $f(2) = \frac{4+e^2}{e^2} \approx 1.541$.



Extra Fun: The Indeterminate Forms 1^∞, ∞^0, and 0^0

Some of the most interesting limits in elementary calculus have the indeterminate forms 1^∞, 0^0, or 0^0. All of these indeterminate limit forms arise from functions that have both a variable base and a variable exponent (power). For example, consider

$$\lim_{x\to 0^+} x^x \qquad \text{Form: } 0^0$$

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x \qquad \text{Form: } 1^\infty$$

$$\lim_{x\to\infty} x^{1/x} \qquad \text{Form: } \infty^0$$

We will use logs and l'Hôpital's rule to simplify some of these limit calculations.

General Form The general form of all of these limits is $\lim_{x\to a}[f(x)]^{g(x)} = y$. To simplify these limits we use the natural log to *undo* the power. If the eventual limit is y (which is unknown to us—it's what we are trying to find, then

$$y = \lim_{x\to a}[f(x)]^{g(x)}.$$

We take the natural log of both sides—here $[f(x)]^{g(x)}$ is assumed to be positive.

$$\ln y = \ln(\lim_{x\to a}[f(x)]^{g(x)})$$

As long as $f(x)$ and $g(x)$ are continuous, we can switch the order of the log and the limit and use log properties

$$\ln y \overset{\text{Cont}}{=} \lim_{x\to a} \ln([f(x)]^{g(x)})$$

$$\ln y = \lim_{x\to a} g(x)\ln(f(x))$$

At this stage we typically use l'Hôpital's rule to find the limit, call it L. Then $\ln y = L$ so we must have $y = e^L$. Let's look at some examples.

EXAMPLE 37.0.12. Determine $\lim_{x\to 0^+}(2x)^x$. Notice that this is a 0^0 form.

SOLUTION. Let $y = \lim_{x\to 0^+} (2x)^x$. We want to find y. Using the log process above,

$$\begin{aligned}
\ln y &= \ln(\lim_{x\to 0^+} (2x)^x) \\
\ln y &\overset{\text{Cont}}{=} \lim_{x\to 0^+} \ln(2x)^x \\
\ln y &= \lim_{x\to 0^+} x\ln 2x \\
\ln y &= \lim_{x\to 0^+} \frac{\ln 2x}{\frac{1}{x}} \\
\ln y &\overset{\text{l'Ho}}{=} \lim_{x\to 0^+} \frac{\frac{2}{2x}}{-\frac{1}{x^2}} \\
\ln y &= \lim_{x\to 0^+} -x \\
\ln y &= 0.
\end{aligned}$$

But $\ln y = 0$ implies $y = e^0 = 1$. So $\lim_{x\to 0^+} (2x)^x = y = 1$. Wow!

EXAMPLE 37.0.13 (Critical Example). Determine $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$. Notice that this is a 1^∞ form.

SOLUTION. Let $y = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$. We want to find y. Using the log process,

$$\begin{aligned}
\ln y &= \ln\left[\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x\right] \\
\ln y &\overset{\text{Cont}}{=} \lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^x \\
\ln y &= \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \\
\ln y &= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \\
\ln y &\overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} \\
\ln y &= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} \\
\ln y &= 1.
\end{aligned}$$

But $\ln y = 1$ implies $y = e^1 = e$. So $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = y = e$. Double Wow!! In fact, in some courses you will see that e is defined this way.

EXAMPLE 37.0.14 (Critical Example). Determine $\lim_{x\to\infty} x^{1/x}$. Notice that this is a ∞^0 form.

SOLUTION. Let $y = \lim_{x\to\infty} x^{1/x}$. We want to find y. Using the log process,

$$\begin{aligned}
\ln y &= \ln \lim_{x\to\infty} x^{1/x} \\
\ln y &\overset{\text{Cont}}{=} \lim_{x\to\infty} \ln x^{1/x} \\
\ln y &= \lim_{x\to\infty} \frac{1}{x}\ln x \\
\ln y &= \lim_{x\to\infty} \frac{\ln x}{x} \\
\ln y &\overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} \\
\ln y &= \frac{0}{1} = 0.
\end{aligned}$$

But $\ln y = 0$ implies $y = e^0 = 1$. So $\lim_{x\to\infty} x^{1/x} = y = 1$. Neat!