## Evaluating Limits Using l'Hôpital's Rule

Some useful limits. Before we look at any further examples and techniques for computing limits, here are some very handy limits that you should know. All of these limits come from looking at the graphs of the particular log or exponential function.

THEOREM 37.0.1 (The End Behavior of the Natural Log and Exponential Functions). The end behavior of $e^{x}$ and $e^{-x}$ on $\left.9-\infty, \infty\right)$ and $\ln x$ on $(0,+\infty)$ is given by

$$
\begin{array}{rll}
\lim _{x \rightarrow \infty} e^{x}=+\infty & \text { and } & \lim _{x \rightarrow-\infty} e^{x}=0 \\
\lim _{x \rightarrow \infty} e^{-x}=0 & \text { and } & \lim _{x \rightarrow-\infty} e^{-x}=+\infty \\
\lim _{x \rightarrow 0^{+}} \ln x=-\infty & \text { and } & \lim _{x \rightarrow \infty} \ln x=+\infty
\end{array}
$$

$f(x)=e^{-x}$



Figure 37.3: The graph of $f(x)=\ln x$.
$\lim _{x \rightarrow \infty} \ln x=+\infty$ and $\lim _{x \rightarrow 0^{+}} \ln x=-\infty$
and $x=0$ is VA for $y=\ln x$.
Figure 37.2: The graph of $f(x)=e^{-x}$.
$\lim _{x \rightarrow \infty} e^{-x}=0$ and $\lim _{x \rightarrow-\infty} e^{-x}=+\infty$. $y=0$ is HA for $y=e^{-x}$.

## Introduction: Indeterminate Forms

Most of the interesting limits in Calculus I have the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Remember that we say that such limits have indeterminate form.

Such limits require "more work" to evaluate them. This work might be factoring, using conjugates, using known limits, or dividing by the highest power of $x$. Here are three common types of indeterminate limits:

1. $\frac{0}{0}$ form: If we let $x=2$ in

$$
\lim _{x \rightarrow 2} \frac{x^{2}-4}{x-2}
$$

we obtain a meaningless $\frac{0}{0}$ expression.
2. $\frac{\infty}{\infty}$ form: If we let $x \rightarrow \infty$ in

$$
\lim _{x \rightarrow \infty} \frac{2 x^{2}-4}{3 x^{2}+9}
$$

we obtain a meaningless $\frac{\infty}{\infty}$ expression.
3. and a new type of indeterminate form $\infty \cdot 0$ : If we let $x=\infty$ in

$$
\lim _{x \rightarrow \infty} x e^{-x}
$$

we end up with a meaningless $\infty \cdot 0$ expression. Each of these limits requires 'more work' to evaluate them, such as factoring or focusing on highest powers. Now we describe a simple method called l'Hôpital's Rule to evaluate limits, at least limits of the first two types.

## The Indeterminate Form $\frac{0}{0}$.

THEOREM 37.0.2 (l'Hôpital's Rule). Let $f$ and $g$ be differentiable on an open interval $I$ containing $a$ with $g^{\prime}(x) \neq 0$ on $I$ when $x \neq a$. If $\lim _{x \rightarrow a} f(x)=0$ and $\lim _{x \rightarrow a} g(x)=0$, then

$$
\lim _{x \rightarrow a} \frac{f(x)}{g(x)}=\lim _{x \rightarrow a} \frac{f^{\prime}(x)}{g^{\prime}(x)}
$$

provided the limit on the right side exists or is $\pm \infty$. This also applies to one-sided limits and to limits as $x \rightarrow \infty$ or $x \rightarrow-\infty$

EXAMPLE 37.0.3. We could evaluate the following indeterminate limit by factoring:

$$
\lim _{x \rightarrow 2} \frac{x^{2}-4^{\nearrow_{0}}}{x-2_{\searrow 0}}=\lim _{x \rightarrow 2} \frac{(x-2)(x+2)}{x-2}=\lim _{x \rightarrow 2} x+2=4 .
$$

But we could also use l'Hôpital's rule:

$$
\lim _{x \rightarrow 2} \frac{x^{2}-4}{x-2} \stackrel{l^{\prime} \mathrm{Ho}}{=} \lim _{x \rightarrow 2} \frac{2 x}{1}=4
$$

which is pretty easy. Just remember to take the derivatives of the numerator and denominator separately.

EXAMPLE 37.0.4. This technique can be applied to problems where our old techniques failed. Here are a few more.

1. $\lim _{x \rightarrow 1} \frac{1-x^{7_{0}}}{\ln x_{\searrow 0}} \stackrel{l^{\prime} \mathrm{Ho}}{=} \lim _{x \rightarrow 1} \frac{-1}{\frac{1}{x}}=\lim _{x \rightarrow 1}-x=-1$
2. $\lim _{x \rightarrow 1} \frac{4^{x}-2^{x}-2^{\text {/0 }}}{x-1}{ }^{\prime}$ 'Ho $\lim _{x \rightarrow 1} \frac{4^{x} \ln 4-2^{x} \ln 2}{1}=4 \ln 4-2 \ln 2$
3. $\lim _{x \rightarrow 0} \frac{1-\cos 3 x^{\nearrow_{0}}}{2 x_{\searrow_{0}}^{2}} \stackrel{l^{\prime} \text { Ho }}{=} \lim _{x \rightarrow 0} \frac{3 \sin 3 x^{\nearrow^{0}}}{4 x_{\searrow 0}}=\lim _{x \rightarrow 0} \frac{9 \cos 3 x}{4}=\frac{9}{4}$
4. $\lim _{x \rightarrow 0} \frac{x^{2}+x^{\nearrow_{0}}}{e^{x}-1}{ }^{\prime}{ }^{\prime}$ Ho $\lim _{x \rightarrow 0} \frac{2 x+1}{e^{x}}=\frac{2}{1}=2$
5. $\lim _{x \rightarrow 0} \frac{x^{2}+x^{\nmid 0}}{e_{\searrow 1}^{x}}=0$. Here l'Hôpital's rule does not apply. The limit can be evaluated since the denominator is not approaching 0 .
6. $\lim _{x \rightarrow 3^{+}} \frac{x-3^{\text {(0 }}}{\ln (2 x-5)_{\searrow 0}} \stackrel{1^{\prime} \text { но }}{=} \lim _{x \rightarrow 3^{+}} \frac{1}{\frac{2}{2 x-5}}=\lim _{x \rightarrow 3^{+}} \frac{2 x-5}{2}=\frac{1}{2}$.
7. $\lim _{x \rightarrow 5} \frac{\sqrt{10+3 x}-5^{70}}{x-5} \stackrel{l^{\prime} \text { Но }}{=} \lim _{x \rightarrow 5} \frac{\frac{3}{2 \sqrt{10+3 x}}}{1}=\frac{3}{10}$.

## Why Should l'Hôpital's Rule Be True?

Here's a proof of a simpler version of l'Hôpital's rule. It makes use of the definition of the derivative.

THEOREM 37.0.5 (l'Hôpital's Rule-Simple Version). Let $f$ and $g$ be differentiable on an open interval $I$ containing $a$ with $g^{\prime}(a) \neq 0$. Assume that $\lim _{x \rightarrow a} f(x)$ and $\lim _{x \rightarrow a} f(x)$ both equal 0 . Then

$$
\lim _{x \rightarrow a} \frac{f(x)}{g(x)}=\frac{f^{\prime}(a)}{g^{\prime}(a)}
$$

Proof. Since $f$ and $g$ are differentiable at $a$, then each is continuous at $x=a$.
Therefore, by definition of continuity, $f(a)=\lim _{x \rightarrow a} f(x)=0$ and $g(a)=\lim _{x \rightarrow a} g(x)=$ 0.

Using the definition of the derivative and that fact that $f(a)=g(a)=0$, we get

$$
\lim _{x \rightarrow a} \frac{f(x)}{g(x)}=\lim _{x \rightarrow a} \frac{\frac{f(x)}{x-a}}{\frac{g(x)}{x-a}}=\lim _{x \rightarrow a} \frac{\frac{f(x)-0}{x-a}}{\frac{g(x)-0}{x-a}}=\lim _{x \rightarrow a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}=\frac{f^{\prime}(a)}{g^{\prime}(a)}
$$

That was easy!

