

### Extra Fun: The Indeterminate Forms $1^\infty$ , $\infty^0$ , and $0^0$

Some of the most interesting limits in elementary calculus have the indeterminate forms  $1^\infty$ ,  $0^0$ , or  $\infty^0$ . All of these indeterminate limit forms arise from functions that have both a variable base and a variable exponent (power). For example, consider

$$\begin{aligned} \lim_{x \rightarrow 0^+} x^x &\quad \text{Form: } 0^0 \\ \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x &\quad \text{Form: } 1^\infty \\ \lim_{x \rightarrow \infty} x^{1/x} &\quad \text{Form: } \infty^0 \end{aligned}$$

We will use logs and l'Hôpital's rule to simplify some of these limit calculations.

*General Form* The general form of all of these limits is  $\lim_{x \rightarrow a} [f(x)]^{g(x)} = y$ . To simplify these limits we use the natural log to *undo* the power. If the eventual limit is  $y$  (which is unknown to us—it's what we are trying to find), then

$$y = \lim_{x \rightarrow a} [f(x)]^{g(x)}.$$

We take the natural log of both sides—here  $[f(x)]^{g(x)}$  is assumed to be positive.

$$\ln y = \ln(\lim_{x \rightarrow a} [f(x)]^{g(x)})$$

As long as  $f(x)$  and  $g(x)$  are continuous, we can switch the order of the log and the limit and use log properties

$$\begin{aligned} \ln y &\stackrel{\text{Cont}}{=} \lim_{x \rightarrow a} \ln([f(x)]^{g(x)}) \\ \ln y &= \lim_{x \rightarrow a} g(x) \ln(f(x)) \end{aligned}$$

At this stage we typically use l'Hôpital's rule to find the limit, call it  $L$ . Then  $\ln y = L$  so we must have  $y = e^L$ . Let's look at some examples.

**EXAMPLE 38.1.1.** Determine  $\lim_{x \rightarrow 0^+} (2x)^x$ . Notice that this is a  $0^0$  form.

**SOLUTION.** Let  $y = \lim_{x \rightarrow 0^+} (2x)^x$ . We want to find  $y$ . Using the log process above,

$$\begin{aligned} \ln y &= \ln\left(\lim_{x \rightarrow 0^+} (2x)^x\right) \\ \ln y &\stackrel{\text{Cont}}{=} \lim_{x \rightarrow 0^+} \ln(2x)^x \\ \ln y &= \lim_{x \rightarrow 0^+} x \ln 2x \\ \ln y &= \lim_{x \rightarrow 0^+} \frac{\ln 2x}{\frac{1}{x}} \\ \ln y &\stackrel{\text{l'Ho}}{=} \lim_{x \rightarrow 0^+} \frac{\frac{2}{2x}}{-\frac{1}{x^2}} \\ \ln y &= \lim_{x \rightarrow 0^+} -x \\ \ln y &= 0. \end{aligned}$$

But  $\ln y = 0$  implies  $y = e^0 = 1$ . So  $\lim_{x \rightarrow 0^+} (2x)^x = y = 1$ . Wow!

**EXAMPLE 38.1.2 (Critical Example).** Determine  $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$ . Notice that this is a  $1^\infty$  form.

**SOLUTION.** Let  $y = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$ . We want to find  $y$ . Using the log process,

$$\begin{aligned}\ln y &= \ln \left[ \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x \right] \\ \ln y &\stackrel{\text{Cont}}{=} \lim_{x \rightarrow \infty} \ln \left(1 + \frac{1}{x}\right)^x \\ \ln y &= \lim_{x \rightarrow \infty} x \ln \left(1 + \frac{1}{x}\right) \\ \ln y &= \lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{1}{x}\right)}{\frac{1}{x}} \\ \ln y &\stackrel{\text{l'Ho}}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} \\ \ln y &= \lim_{x \rightarrow \infty} \frac{1}{1 + \frac{1}{x}} \\ \ln y &= 1.\end{aligned}$$

But  $\ln y = 1$  implies  $y = e^1 = e$ . So  $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = y = e$ . Double Wow!! In fact, in some courses you will see that  $e$  is defined this way.

**EXAMPLE 38.1.3** (Critical Example). Determine  $\lim_{x \rightarrow \infty} x^{1/x}$ . Notice that this is a  $\infty^0$  form.

**SOLUTION.** Let  $y = \lim_{x \rightarrow \infty} x^{1/x}$ . We want to find  $y$ . Using the log process,

$$\begin{aligned}\ln y &= \ln \lim_{x \rightarrow \infty} x^{1/x} \\ \ln y &\stackrel{\text{Cont}}{=} \lim_{x \rightarrow \infty} \ln x^{1/x} \\ \ln y &= \lim_{x \rightarrow \infty} \frac{1}{x} \ln x \\ \ln y &= \lim_{x \rightarrow \infty} \frac{\ln x}{x} \\ \ln y &\stackrel{\text{l'Ho}}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} \\ \ln y &= \frac{0}{1} = 0.\end{aligned}$$

But  $\ln y = 0$  implies  $y = e^0 = 1$ . So  $\lim_{x \rightarrow \infty} x^{1/x} = y = 1$ . Neat!

**EXAMPLE 38.1.4.** Determine  $\lim_{x \rightarrow \infty} (e^x + x)^{2/x}$ . Notice that this is a  $\infty^0$  form again.

**SOLUTION.** As usual let  $y = \lim_{x \rightarrow \infty} (e^x + x)^{2/x}$ , so

$$\begin{aligned}\ln y &= \ln \lim_{x \rightarrow \infty} (e^x + x)^{2/x} \stackrel{\text{Cont}}{=} \lim_{x \rightarrow \infty} \ln (e^x + x)^{2/x} \\ \ln y &= \lim_{x \rightarrow \infty} \frac{2}{x} \ln(e^x + x) \\ \ln y &= \lim_{x \rightarrow \infty} \frac{2 \ln(e^x + x)}{x} \\ \ln y &\stackrel{\text{l'Ho}}{=} \lim_{x \rightarrow \infty} 2 \frac{\frac{e^x+1}{e^x+x}}{1} \\ \ln y &= \lim_{x \rightarrow \infty} 2 \frac{e^x+1}{e^x+x} \\ \ln y &\stackrel{\text{l'Ho}}{=} \lim_{x \rightarrow \infty} 2 \frac{e^x}{e^x+1} \\ \ln y &\stackrel{\text{l'Ho}}{=} \lim_{x \rightarrow \infty} 2 \frac{e^x}{e^x} \\ \ln y &= 2.\end{aligned}$$

But  $\ln y = 2$  implies  $y = e^2$ . So  $\lim_{x \rightarrow \infty} (e^x + x)^{2/x} = e^2$ .

**EXAMPLE 38.1.5.** Determine  $\lim_{x \rightarrow 2^+} [5(x - 2)]^{x-2}$ . Notice that this is a  $0^0$  form.

**SOLUTION.** Let  $y = \lim_{x \rightarrow 2^+} [5(x - 2)]^{x-2}$ . We want to find  $y$ . Using the log process above,

$$\begin{aligned}\ln y &= \ln\left(\lim_{x \rightarrow 2^+} [5(x - 2)]^{x-2}\right) \\ \ln y &\stackrel{\text{Cont}}{=} \lim_{x \rightarrow 2^+} \ln[5(x - 2)]^{x-2} \\ \ln y &= \lim_{x \rightarrow 2^+} (x - 2) \ln[5(x - 2)] \\ \ln y &= \lim_{x \rightarrow 2^+} \frac{[5(x - 2)]}{\frac{1}{x-2}} \\ \ln y &\stackrel{\text{l'Ho}}{=} \lim_{x \rightarrow 2^+} \frac{\frac{5}{x-2}}{-\frac{1}{(x-2)^2}} \\ \ln y &= \lim_{x \rightarrow 2^+} -5(x - 2) \\ \ln y &= 0.\end{aligned}$$

But  $\ln y = 0$  implies  $y = e^0 = 1$ . So  $\lim_{x \rightarrow 2^+} [5(x - 2)]^{x-2} = y = 1$ . This is becoming routine.

**YOU TRY IT 38.3.** Here's a great problem to see if you have mastered these ideas. Determine  $\lim_{x \rightarrow 0^+} [\sin(x)]^x$ .

Answer to **YOU TRY IT 38.3**: 1. Hint: Use l'Hôpital's rule twice.

### Problems

1. Some interesting limits. Answers not in order: 0, 0, 0,  $\frac{1}{2}$ ,  $\ln 4$ , 1, 1, 2,  $e^2$ ,  $e^k$ ,  $+\infty$ ,  $-\infty$ , and -6,

- |   |   |  |
|---|---|--|
| (a) $\lim_{x \rightarrow 0} \frac{\cos 4x - \cos 2x}{x^2}$    | (b) $\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2}$  | (c) $\lim_{x \rightarrow 0^+} 2x \ln x$                          |
| (d) $\lim_{x \rightarrow \infty} x^2 e^{-x}$                  | (e) $\lim_{x \rightarrow 0^+} \frac{\cos x}{x^2}$     | (f) $\lim_{x \rightarrow \infty} [\ln(4x + 9) - \ln(x + 7)]$     |
| (g) $\lim_{x \rightarrow 0^+} (3x)^x$                         | (h) $\lim_{x \rightarrow 0^+} (1 + 2x)^{1/x}$         | (i) $\lim_{x \rightarrow \infty} \left(1 + \frac{k}{x}\right)^x$ |
| (j) $\lim_{x \rightarrow 0} \frac{\arctan 4x}{\sin 2x}$       | (k) $\lim_{x \rightarrow 0} \frac{\sin 4x}{3 \sec x}$ | (l) $\lim_{x \rightarrow 1^+} \frac{x^2 + 1}{1 - x}$             |
| (m) $\lim_{x \rightarrow 0^+} \left(\frac{1}{x}\right)^{x^2}$ |   |  |

*Solutions*

1. Make sure to check those stages at which l'Hôpital's rule applies.

$$(a) \lim_{x \rightarrow 0} \frac{\cos 4x - \cos 2x}{x^2} \stackrel{l'H}{=} \lim_{x \rightarrow 0} \frac{-4 \sin 4x + 2 \sin 2x}{2x} \stackrel{l'H}{=} \lim_{x \rightarrow 0} \frac{-16 \cos 4x + 4 \cos 2x}{2} =$$

$$\frac{-16 + 4}{2} = -6.$$

$$(b) \lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2} \stackrel{l'H}{=} \lim_{x \rightarrow 0} \frac{e^x - 1}{2x} \stackrel{l'H}{=} \lim_{x \rightarrow 0} \frac{e^x}{2} = \frac{1}{2}.$$

$$(c) \lim_{x \rightarrow 0^+} 2x \ln x = \lim_{x \rightarrow 0^+} \frac{2 \ln x}{\frac{1}{x}} \stackrel{l'H}{=} \lim_{x \rightarrow 0^+} \frac{\frac{2}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} -\frac{2x^2}{x} = \lim_{x \rightarrow 0^+} -2x = 0.$$

$$(d) \lim_{x \rightarrow \infty} x^2 e^{-x} = \lim_{x \rightarrow \infty} \frac{x^2}{e^x} \stackrel{l'H}{=} \lim_{x \rightarrow \infty} \frac{2x}{e^x} \stackrel{l'H}{=} \lim_{x \rightarrow \infty} \frac{2}{e^x} = 0 \text{ (i.e., } \rightarrow \frac{2}{\infty})$$

$$(e) \lim_{x \rightarrow 0^+} \frac{\cos x}{x^2} \rightarrow \frac{1}{0^+} : +\infty. \text{ l'Hôpital's rule does not apply.}$$

$$(f) \lim_{x \rightarrow \infty} \ln(4x + 9) - \ln(x + 7) = \lim_{x \rightarrow \infty} \ln \left( \frac{4x + 9}{x + 7} \right) = \ln \left( \lim_{x \rightarrow \infty} \frac{4x + 9}{x + 7} \right) \stackrel{l'H}{=} \ln \lim_{x \rightarrow \infty} \frac{4}{1} \\ = \ln 4.$$

(g) Let  $y = \lim_{x \rightarrow 0^+} (3x)^x$ . We want to find  $y$ . Using the log process,

$$\ln = \ln(\lim_{x \rightarrow 0^+} (3x)^x)$$

$$\ln y \stackrel{\text{Cont}}{=} \lim_{x \rightarrow 0^+} \ln(3x)^x$$

$$\ln y = \lim_{x \rightarrow 0^+} x \ln 3x$$

$$\ln y = \lim_{x \rightarrow 0^+} \frac{\ln 3x}{\frac{1}{x}}$$

$$\ln y \stackrel{l'H}{=} \lim_{x \rightarrow 0^+} \frac{\frac{3}{3x}}{-\frac{1}{x^2}}$$

$$\ln y = \lim_{x \rightarrow 0^+} -x$$

$$\ln y = 0.$$

But  $\ln y = 0$  implies  $y = e^0 = 1$ . So  $\lim_{x \rightarrow 0^+} (3x)^x = y = 1$ .

(h) "1 $^\infty$ ": Let  $y = \lim_{x \rightarrow 0^+} (1 + 2x)^{1/x}$ , so

$$\ln y = \ln \lim_{x \rightarrow 0^+} (1 + 2x)^{1/x}$$

$$\ln y \stackrel{\text{Cont}}{=} \lim_{x \rightarrow 0^+} \ln(1 + 2x)^{1/x}$$

$$\ln y = \lim_{x \rightarrow 0^+} \frac{1}{x} \ln(1 + 2x)$$

$$\ln y = \lim_{x \rightarrow 0^+} \frac{\ln(1 + 2x)}{x}$$

$$\ln y \stackrel{l'H}{=} \lim_{x \rightarrow 0^+} \frac{\frac{2}{1+2x}}{1}$$

$$\ln y = \frac{2}{1}$$

$$\ln y = 2.$$

But  $\ln y = 2$  implies  $y = e^2$ . So  $\lim_{x \rightarrow 0^+} (1 + 2x)^{1/x} = e^2$ .

(i) "1 $^\infty$ ": Let  $y = \lim_{x \rightarrow 0^+} (1 + kx)^{1/x}$ , so

$$\begin{aligned}
 \ln y &= \ln \lim_{x \rightarrow 0^+} (1 + kx)^{1/x} \\
 \ln y &\stackrel{\text{Cont}}{=} \lim_{x \rightarrow 0^+} \ln(1 + kx)^{1/x} \\
 \ln y &= \lim_{x \rightarrow 0^+} \frac{1}{x} \ln(1 + kx) \\
 \ln y &= \lim_{x \rightarrow 0^+} \frac{\ln(1 + kx)}{x} \\
 \ln y &\stackrel{\text{l'Ho}}{=} \lim_{x \rightarrow 0^+} \frac{\frac{k}{1+kx}}{1} \\
 \ln y &= \frac{k}{1} \\
 \ln y &= k.
 \end{aligned}$$

But  $\ln y = k$  implies  $y = e^k$ . So  $\lim_{x \rightarrow 0^+} (1 + kx)^{1/x} = e^k$ .

$$(j) \lim_{x \rightarrow 0} \frac{\arctan 4x}{\sin 2x} \stackrel{\text{l'Ho}}{=} \lim_{x \rightarrow 0} \frac{\frac{4}{1+16x^2}}{2 \cos 2x} = \frac{\frac{4}{1}}{2} = 2.$$

$$(k) \lim_{x \rightarrow 0} \frac{\sin 4x}{3 \sec x} = \frac{0}{3} = 0. \text{ l'Hôpital's rule does not apply.}$$

$$(l) \lim_{x \rightarrow 1^+} \frac{x^2 + 1}{1 - x} \rightarrow \frac{2}{0^-} : -\infty. \text{ l'Hôpital's rule does not apply.}$$

(m) " $\infty^0$ ": Let  $y = \lim_{x \rightarrow 0^+} \left(\frac{1}{x}\right)^{x^2}$ , so

$$\begin{aligned}
 \ln y &= \ln \lim_{x \rightarrow 0^+} \left(\frac{1}{x}\right)^{x^2} \\
 \ln y &\stackrel{\text{Cont}}{=} \lim_{x \rightarrow 0^+} \ln \left(\frac{1}{x}\right)^{x^2} \\
 \ln y &= \lim_{x \rightarrow 0^+} x^2 \ln \left(\frac{1}{x}\right) \\
 \ln y &= \lim_{x \rightarrow 0^+} \frac{-\ln x}{\frac{1}{x^2}} \\
 \ln y &\stackrel{\text{l'Ho}}{=} \lim_{x \rightarrow 0^+} \frac{-\frac{1}{x}}{-\frac{2}{x^3}} \\
 \ln y &= \lim_{x \rightarrow 0^+} \frac{x^3}{2x} \\
 \ln y &= \lim_{x \rightarrow 0^+} \frac{x^2}{2} \\
 \ln y &= 0.
 \end{aligned}$$

But  $\ln y = 0$  implies  $y = 1$ . So  $y = \lim_{x \rightarrow 0^+} \left(\frac{1}{x}\right)^{x^2} = 1$ .