## Antiderivatives

### 39.1 Introduction

So far much of the term has been spent finding derivatives or rates of change. But in some circumstances we already know the rate of change and we wish to determine the original function. For example, meters or data loggers often measure rates of change, e.g., miles per hour or kilowatts per hour.

If you know the velocity of an object, can you determine the position of the object. This could happen in a car, say, where the speedometer readings were being recorded. Can the position of the car be determined from this information? Similarly, can the position of an airplane be determined from the black box which records the airspeed?

More generally, given $f^{\prime}(x)$ can we find the function $f(x)$. If you think about it, this is the sort of question I have asked you to do on labs, tests, and homework assignments where I gave you the graph of $f^{\prime}(x)$ and said draw the graph of $f(x)$. Or where I gave you the number line information for $f^{\prime}(x)$ and $f^{\prime \prime}(x)$ and asked you to reconstruct the graph of $f(x)$. Can we do this same thing if we start with a formula for $f^{\prime}(x)$ ? Can we get an explicit formula for $f(x)$ ? We usually state the problem this way.

DEFINITION 39.1.1. Let $f(x)$ be a function defined on an interval $I$. We say that $F(x)$ is an antiderivative of $f(x)$ on $I$ if

$$
F^{\prime}(x)=f(x) \quad \text { for all } x \in I
$$

EXAMPLE 39.1.2. If $f(x)=2 x$, then $F(x)=x^{2}$ is an antiderivative of $f$ because

$$
F^{\prime}(x)=2 x=f(x)
$$

But so is $G(x)=x^{2}+1$ or, more generally, $H(x)=x^{2}+c$.
Are there 'other' antiderivatives of $f(x)=2 x$ besides those of the form $H(x)=$ $x^{2}+c$ ? We can use the MVT to show that the answer is 'No.' The proof will require three small steps.

THEOREM 39.1.3 (Theorem 1). If $F^{\prime}(x)=0$ for all $x$ in an interval $I$, then $F(x)=k$ is a constant function.

This makes a lot of sense: If the velocity of an object is o , then its position is constant (not changing). Here's the

Proof. To show that $F(x)$ is constant, we must show that any two output values of $F$ are the same, i.e., $F(a)=F(b)$ for all $a$ and $b$ in $I$.

So pick any $a$ and $b$ in $I$ (with $a<b$ ). Then since $F$ is differentiable on $I$, then $F$ is both continuous and differentiable on the smaller interval $[a, b]$. So the MVT applies. There is a point $c$ between $a$ and $b$ so that

$$
\frac{F(b)-F(a)}{b-a}=F^{\prime}(c)=0
$$

since the derivative of $F$ is always 0 . The only way this fraction can be equal to 0 is if the numerator is $F(b)-F(a)=0$. This means

$$
F(b)=F(a)
$$

in other words, $F$ is constant.

THEOREM 39.1.4 (Theorem 2: Families of Antiderivatives). If $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$ on an interval $I$, then $G(x)=F(x)+k$.

Proof. If $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$ on an interval $I$ then $G^{\prime}(x)=f(x)$ and $F^{\prime}(x)=f(x)$, that is, $F^{\prime}(x)=G^{\prime}(x)$ on $I$.

Now consider the function $G(x)-F(x)$ on $I$.

$$
\frac{d}{d x}(G(x)-F(x))=G^{\prime}(x)-F^{\prime}(x)=0
$$

Therefore, by Theorem 39.1.3, there is a constant $k$ so that

$$
G(x)-F(x)=k
$$

or

$$
G(x)=F(x)+k
$$

DEFINITION 39.1.5. If $F(x)$ is any antiderivative of $f(x)$, we say that $F(x)+c$ is the general antiderivative of $f(x)$ on $I$.

## Notation for Antiderivatives

Antidifferentiation is also called 'indefinite integration.'

$$
\int f(x) d x=F(x)+c
$$

- $\int$ is the integration symbol
- $f(x)$ is called the integrand
- $d x$ indicates the variable of integration
- $F(x)$ is a particular antiderivative of $f(x)$
- and $c$ is the constant of integration.
- We refer to $\int f(x) d x$ as an 'antiderivative of $f(x)^{\prime}$ or an 'indefinite integral of f.'

Here are several examples.

$$
\begin{aligned}
\int 2 x d x & =x^{2}+c \\
\int \cos t d t & =\sin t+c \\
\int e^{z} d z & =e^{z}+c \\
\int \frac{1}{1+x^{2}} d x & =\arctan x+c
\end{aligned}
$$

Antidifferentiation reverses differentiation so

$$
\int F^{\prime}(x) d x=F(x)+c
$$

and differentiation undoes antidifferentiation

$$
\frac{d}{d x}\left[\int f(x) d x\right]=f(x)
$$

Differentiation and antidifferentiation are reverse processes, so each derivative rule has a corresponding antidifferentiation rule.

$$
\begin{aligned}
& \text { Differentiation } \\
& \frac{d}{d x}(c)=0 \\
& \frac{d}{d x}(k x)=k \\
& \frac{d}{d x}\left(x^{n}\right)=n x^{n-1} \\
& \frac{d}{d x}(\ln |x|)=\frac{1}{x} \\
& \frac{d}{d x}(\sin x)=\cos x \\
& \frac{d}{d x}(\cos x)=-\sin x \\
& \frac{d}{d x}(\tan x)=\sec ^{2} x \\
& \frac{d}{d x}(\sec x)=\sec x \tan x \\
& \frac{d}{d x}\left(e^{x}\right)=e^{x} \\
& \frac{d}{d x}(\arcsin x)=\frac{1}{\sqrt{1-x^{2}}} \\
& \frac{d}{d x}(\arctan x)=\frac{1}{1+x^{2}}
\end{aligned}
$$

## Antidifferentiation

$\int 0 d x=c$
$\int k d x=k x+c$
$\int x^{n} d x=\frac{x^{n+1}}{n+1}+c, n \neq-1$
$\int x^{-1} d x=\int \frac{1}{x} d x=\ln |x|+c$
$\int \cos x d x=\sin x+c$
$\int \sin x d x=-\cos x+c$
$\int \sec ^{2} x d x=\tan x+c$
$\int \sec x \tan x d x=\sec x+c$
$\int e^{x} d x=e^{x}+c$
$\int \frac{1}{\sqrt{1-x^{2}}} d x=\arcsin x+c$
$\int \frac{1}{1+x^{2}} d x=\arctan x+c$

## Variations and Generalizations

Notice what happens when we use $a x$ instead of $x$ in some of these functions. We multiply by $a$ when taking the derivative, so we have to divide by $a$ when taking the antiderivative.

## Differentiation

$$
\begin{array}{ll}
\frac{d}{d x}\left(e^{a x}\right)=a e^{a x} & \int e^{a x} d x=\frac{1}{a} e^{a x}+c \\
\frac{d}{d x}(\sin a x)=a \cos a x & \int \cos a x d x=\frac{1}{a} \sin a x+c \\
\frac{d}{d x}(\tan a x)=a \sec ^{2} a x & \int \sec ^{2} a x d x=\frac{1}{a} \tan a x+c \\
\frac{d}{d x}\left(\arcsin \left(\frac{x}{a}\right)=\frac{1}{\sqrt{a^{2}-x^{2}}}\right. & \int \frac{1}{\sqrt{a^{2}-x^{2}}} d x=\arcsin \left(\frac{x}{a}\right)+c \\
\frac{d}{d x}\left(\arctan \left(\frac{x}{a}\right)\right)=\frac{a}{a^{2}+x^{2}} & \int \frac{1}{a^{2}+x^{2}} d x=\frac{1}{a} \arctan \left(\frac{x}{a}\right)+c
\end{array}
$$

Try filling in the rules for $\int \sin a x d x$ and $\int \sec (a x) \tan (a x) d x$.
EXAMPLE 39.1.6. Here are a few examples.

$$
\begin{aligned}
\int \cos (4 x) d x & =\frac{1}{4} \sin (4 x)+c \\
\int e^{z / 2} d z & =2 e^{z / 2}+c \\
\int \frac{1}{16+x^{2}} d x & =\frac{1}{4} \arctan \frac{x}{4}+c
\end{aligned}
$$

### 39.2 Problems

1. Determine antiderivatives of the following functions. Take the derivative of your answer to confirm that you are correct. Why should you add $+c$ to any answer? Basics:
(a) $7 x^{6}$
(b) $x^{6}$
(c) $2 x^{6}$
(d) $e^{x}$
(e) $2 e^{x}$
(f) $2 e^{2 x}$
(g) $e^{2 x}$
(h) $\frac{1}{x}$
(i) $-\frac{8}{x}$
(j) $\cos x$
(k) $4 \cos x$
(l) $4 \cos (4 x)$
(m) $\cos (4 x)$
(n) $-\frac{1}{x}+4 \cos x$
(o) $-2 \sec x \tan x$
(p) $\sin 4 x$
(q) $4 \sec ^{2} x$
(r) $2 x$
(s) $x$
(t) $8 x$
(u) $6^{x} \ln 6$
(v) $6 x-6^{x} \ln 6$
(w) 6
(x) $6+\sec ^{2} x$
(y) $\frac{6}{1+x^{2}}$
(z) $\frac{2}{\sqrt{1-x^{2}}}$
2. Now try these. Think it through.
(a) $x^{8}+2$
(b) $e^{8 x}-c$
(c) $-2 \sin 4 x$
(d) $\cos 2 \theta-\sec ^{2} 2 \theta$
(e) $x^{2}+3 x-1$
(f) $\frac{\cos x}{2}$
(g) $x^{5 / 2}$
(h) $x^{-3 / 5}$
(i) $\frac{1}{\sqrt[4]{x^{7}}}$
(j) $\frac{4 x^{3}}{x^{4}+12}$
(k) $14 x\left(x^{2}+5\right)^{6}$
(l) $e^{x} \cos \left(e^{x}\right)$
(m) $12 x e^{x^{2}+9}$
(n) $\frac{12(\ln x)^{5}}{x}$
3. Answers
(a) $x^{7}+c$
(b) $\frac{x^{7}}{7}+c$
(c) $\frac{2}{7} x^{7}+c$
(d) $e^{x}+c$
(e) $2 e^{x}+c$
(f) $e^{2 x}+c$
(g) $\frac{1}{2} e^{2 x}+c$
(h) $\ln |x|+c$
(i) $-8 \ln |x|+c$
(j) $\sin x+c$
(k) $4 \sin x+c$
(l) $\sin (4 x)+c$
(m) $\frac{1}{4} \sin (4 x)+c$
(n) $-\ln |x|+4 \sin x+c$
(o) $-2 \sec x+c$
(p) $-\frac{1}{4} \cos 4 x+c$
(q) $4 \tan x+c$
(r) $x^{2}+c$
(s) $\frac{1}{2} x^{2}+c$
(t) $4 x^{2}+c$
(u) $6^{x}+c$
(v) $3 x^{2}-6^{x}+c$
(w) $6 x+c$
(x) $6 x+\tan x+c$
(y) $6 \arctan (x)+c$
(z) $2 \arcsin (x)+c$
4. Answers.
(a) $\frac{1}{9} x^{9}+2 x+c$
(b) $\frac{1}{8} e^{8 x}-c x+d$
(c) $\frac{1}{2} \cos 4 x+c$
(d) $\frac{1}{2} \sin 2 \theta-\frac{1}{2} \tan 2 \theta+c$
(e) $\frac{1}{3} x^{3}+\frac{3}{2} x^{2}-x+c$
(f) $\frac{\sin x}{2}+c$
(g) $\frac{2}{7} x^{7 / 2}+c$
(h) $\frac{5}{2} x^{2 / 5}+c$
(i) $-\frac{4}{3} x^{-3 / 4}$
(j) $\ln \left(x^{4}+12\right)+c$
(k) $\left(x^{2}+5\right)^{7}+c$
(l) $\sin \left(e^{x}\right)+c$
(m) $6 e^{x^{2}+9}+c$
(n) $2(\ln x)^{6}+c$

### 40.3 General Antiderivative Rules

The key idea is that each derivative rule can be written as an antiderivative rule.
We've seen how this works with specific functions like $\sin x$ and $e^{x}$ and now we examine how the general derivative rules can be 'reversed.'

FACT 40.1 (Sum Rule). The sum rule for derivatives says

$$
\frac{d}{d x}(F(x) \pm G(x))=\frac{d}{d x}(F(x)) \pm \frac{d}{d x}(G(x)) .
$$

The corresponding antiderivative rule is

$$
\int(f(x) \pm g(x)) d x=\int f(x) d x \pm \int g(x) d x
$$

FACT 40.2 (Constant Multiple Rule). The constant multiple rule for derivatives says

$$
\frac{d}{d x}(c F(x))=c \frac{d}{d x}(F(x)) .
$$

The corresponding antiderivative rule is

$$
\int c f(x) d x=c \int f(x) d x
$$

## Examples

$\int 8 x^{3}-7 \sqrt{x} d x=\int 8 x^{3} d x-\int 7 x^{1 / 2} d x=8 \int x^{3} d x-7 \int x^{1 / 2} d x=\frac{8 x^{4}}{4}-\frac{7 x^{3 / 2}}{3 / 2}+c=2 x^{4}-\frac{14 x^{3 / 2}}{3}+c$
$\int 6 \cos 2 x-\frac{7}{x}+2 x^{-1 / 3} d x=6 \int \cos 2 x d x-7 \int \frac{1}{x} d x+2 \int x^{-1 / 3} d x=6 \cdot \frac{1}{2} \cdot \sin 2 x-7 \ln |x|+\frac{2 x^{2 / 3}}{2 / 3}+c=3 \sin 2 x-7 \ln |x|+3 x^{2 / 3}+c$.
$\int 3 e^{x / 2}-\frac{8}{\sqrt{9-x^{2}}} d x=3 \int e^{x / 2} d x-8 \int \frac{1}{\sqrt{9-x^{2}}} d x=3 \cdot 2 e^{x / 2}+8 \arcsin \frac{x}{3}+c .=6 e^{x / 2}+8 \arcsin \frac{x}{3}+c$.

## Rewriting Integrals

Rewriting the integrand can greatly simplify the antiderivative process.
EXAMPLE 40.3.1. Here are several examples.
(a) Determine $\int 2 \sqrt[5]{t^{2}}-6 \sec ^{2} t d t$

SOLUTION. Rewrite using fractional exponents.

$$
\int 2 \sqrt[5]{t^{2}}-6 \sec ^{2} t d t=\int 2 t^{2 / 5}-6 \sec ^{2} t d t=\frac{2 t^{7 / 5}}{7 / 5}-6 \tan t+c=\frac{10 t^{7 / 5}}{7}-6 \tan t+c
$$

(b) Determine $\int \frac{x^{4}+2}{x^{2}} d x$.

SOLUTION. Divide first to simplify.

$$
\int \frac{x^{4}+2}{x^{2}} d x=\int x^{2}+2 x^{-2} d x=\frac{x^{3}}{3}+\frac{2 x^{-1}}{-1}+c=\frac{x^{3}}{3}-2 x^{-1}+c
$$

(c) Determine $\int 6 x^{2}\left(x^{4}-1\right) d x$.

SOLUTION. Multiply out to obtain a polynomial.

$$
\int 6 x^{2}\left(x^{4}-1\right) d x=\int 6 x^{6}-6 x^{2} d x=\frac{6 x^{7}}{7}+\frac{6 x^{3}}{3}+c=\frac{6 x^{7}}{7}+2 x^{3}+c
$$

(d) Determine $\int \frac{2}{\sqrt[3]{t^{5}}} d t$

SOLUTION. Rewrite using fractional exponents.

$$
\int \frac{2}{\sqrt[3]{t^{5}}} d t=\int 2 t^{-5 / 3} d t=\frac{2 t^{-2 / 3}}{-2 / 3}+c=-3 t^{-2 / 3}+c
$$

(e) Determine $\int \frac{8 x^{2}+7}{\sqrt{x}} d x$
solution. Divide first to simplify.

$$
\int \frac{8 x^{2}+7}{\sqrt{x}} d x=\int 8 x^{3 / 2}+7 x^{-1 / 2} d x=\frac{8 x^{5 / 3}}{5 / 3}+\frac{7 x^{1 / 2}}{1 / 2}+c=\frac{24 x^{5 / 3}}{5}+14 x^{1 / 2}+c .
$$

