## Initial Value Problems

### 40.1 Evaluating ' ${ }^{\prime}$ '

So far we have been calculating general antiderivatives of functions. What this means is that if we know the velocity $v(t)$ of the car we are driving in, we can determine the position $p(t)$ of the car, up to a constant $c$ if we can find an antiderivative for $v(t)$. If we have more information, the position of the car at a particular time say, then we are able to determine the precise antiderivative.
EXAMPLE 40.1.1. Suppose that $f^{\prime}(x)=e^{x}+2 x$ and $f(0)=3$. Find $f(x) . f(0)$ is sometimes called the initial value and such questions are referred to as initial value problems. [How could you interpret this information in terms of motion?]
Solution. $f(x)$ must be an antiderivative of $f^{\prime}(x)$ so

$$
f(x)=\int f^{\prime}(x) d x=\int e^{x}+2 x d x=e^{x}+x^{2}+c
$$

Now use the initial value to solve for $c$ :

$$
f(0)=e^{0}+0^{2}+c=3 \Rightarrow 1+c=3 \Rightarrow c=2
$$

Therefore, $f(x)=e^{x}+x^{2}+2$.
EXAMPLE 40.1.2. Suppose that $f^{\prime}(x)=6 x^{2}-2 x^{3}$ and $f(1)=4$. Find $f(x)$.
Solution. Again $f(x)$ must be an antiderivative of $f^{\prime}(x)$ so

$$
f(x)=\int f^{\prime}(x) d x=\int 6 x^{2}-2 x^{3} d x=2 x^{3}-\frac{x^{4}}{2}+c
$$

Now use the 'initial' value to solve for $c$ :

$$
f(1)=2-1 / 2+c=4 \Rightarrow c=2.5
$$

Therefore, $f(x)=2 x^{3}-\frac{x^{4}}{2}+2.5$.
EXAMPLE 40.1.3. Suppose that $f^{\prime \prime}(t)=6 t^{-2}$ (think acceleration) with $f^{\prime}(1)=8$ (think velocity) and $f(1)=3$ (think position). Find $f(t)$.
Solution. First find $f^{\prime}(t)$ which must be the antiderivative of $f^{\prime \prime}(t)$. So

$$
f^{\prime}(t)=\int f^{\prime \prime}(t) d t=\int 6 x^{-2} d t=-6 t^{-1}+c
$$

Now use the 'initial' value for $f^{\prime}(t)$ to solve for $c$ :

$$
f^{\prime}(1)=-6(1)+c=8 \Rightarrow c=14
$$

Therefore, $f^{\prime}(t)=-6 t^{-1}+14$. Now we are back to the earlier problem.

$$
f(t)=\int f^{\prime}(t) d t=\int-6 t^{-1}+14 d t=-6 \ln |t|+14 t+c
$$

Now use the 'initial' value of $f$ to solve for $c$ :

$$
f(1)=-6 \ln 1+14(1)+c=3 \Rightarrow 6(0)+14+c=3 \Rightarrow c=-11
$$

So $f(t)=6 \ln |t|+14 t-11$.

## In Class Practice

EXAMPLE 40.1.4. Find $f$ given that $f^{\prime}(x)=6 \sqrt{x}+5 x^{\frac{3}{2}}$ where $f(1)=10$.
Solution. $\quad f(x)$ must be an antiderivative of $f^{\prime}(x)$ so

$$
f(x)=\int f^{\prime}(x) d x=\int 6 \sqrt{x}+5 x^{\frac{3}{2}} d x=4 x^{3 / 2}+2 x^{5 / 2}+c
$$

Use the 'initial' value to solve for $c$ :

$$
f(1)=4+2+c=10 \Rightarrow c=4
$$

Therefore, $f(x)=4 x^{3 / 2}+2 x^{5 / 2}+4$.
EXAMPLE 40.1.5. Find $f$ given that $f^{\prime \prime}(\theta)=\sin \theta+\cos \theta$ where $f^{\prime}(0)=1$ and $f(0)=2$.
Solution. First find $f^{\prime}(\theta)$ which must be the antiderivative of $f^{\prime \prime}(\theta)$. So

$$
f^{\prime}(\theta)=\int f^{\prime \prime}(\theta) d \theta=\int \sin \theta+\cos \theta=-\cos \theta+\sin \theta+c
$$

Now use the initial value for $f^{\prime}(\theta)$ to solve for $c$ :

$$
f^{\prime}(0)=-\cos 0+\sin 0+c=-1+0+c=1 \Rightarrow c=2
$$

Therefore, $f^{\prime}(\theta)=-\cos \theta+\sin \theta+2$.

$$
f(\theta)=\int f^{\prime}(\theta) d \theta=\int-\cos \theta+\sin \theta+2 d \theta=-\sin \theta-\cos \theta+2 \theta+c
$$

Now use the initial value of $f$ to solve for $c$ :

$$
f(0)=-\sin 0-\cos 0+2(0)+c=0-1+c=2 \Rightarrow c=3
$$

So $f(\theta)=-\sin \theta-\cos \theta+2 \theta+3$.

### 40.2 Motion Problems

## Introduction

Earlier in the term we interpreted the first and second derivatives as velocity and acceleration in the context of motion. So let's apply the initial value problem results to motion problems. Recall that

- $s(t)=$ position at time $t$.
- $s^{\prime}(t)=v(t)$ velocity at time $t$.
- $s^{\prime \prime}(t)=v^{\prime}(t)=a(t)$ acceleration at time $t$.

Therefore

- $\int a(t) d t=v(t)+c_{1}$ velocity.
- $\int v(t) d t=s(t)+c_{2}$ position at time $t$.

We will need to use additional information to evaluate the constants $c_{1}$ and $c_{2}$.
EXAMPLE 40.2.1. Suppose that the acceleration of an object is given by $a(t)=2-\cos t$ for $t \geq 0$ with

- $v(0)=1$, this is also denoted $v_{0}$
- $s(0)=3$, this is also denoted $s_{0}$.

Find $s(t)$.
Solution. First find $v(t)$ which is the antiderivative of $a(t)$.

$$
v(t)=\int a(t) d t=\int 2-\cos t d t=2 t-\sin t+c_{1}
$$

Now use the initial value for $v(t)$ to solve for $c_{1}$ :

$$
v(0)=0-0+c_{1}=1 \Rightarrow c_{1}=1
$$

Therefore, $v(t)=2 t-\sin t+1$. Now solve for $s(t)$ by taking the antiderivative of $v(t)$.

$$
s(t)=\int v(t) d t=\int 2 t-\sin t+1 d t=t^{2}+\cos t+t+c_{2}
$$

Now use the initial value of $s$ to solve for $c_{2}$ :

$$
s(0)=0+\cos 0+c=3 \Rightarrow 1+c=3 \Rightarrow c=2
$$

So $s(t)=t^{2}+\cos t+2 t+2$.
EXAMPLE 40.2.2. If acceleration is given by $a(t)=10+3 t-3 t^{2}$, find the position function if $s(0)=1$ and $s(2)=11$.

Solution. First

$$
v(t)=\int a(t) d t=\int 10+3 t-3 t^{2} d t=10 t+\frac{3}{2} t^{2}-t^{3}+c
$$

Now

$$
s(t)=\int 10 t+\frac{3}{2} t^{2}-t^{3}+c d t=5 t^{2}+\frac{1}{2} t^{3}-\frac{1}{4} t^{4}+c t+d
$$

But $s(0)=0+0-0+0+d=1$ so $d=1$. Then $s(2)=20+4-4+2 c+1=11$ so $2 c=-10 \Rightarrow c=-5$. Thus, $s(t)=5 t^{2}+\frac{1}{2} t^{3}-\frac{1}{4} t^{4}-5 t+1$.
EXAMPLE 40.2.3. If acceleration is given by $a(t)=\sin t+\cos t$, find the position function if $s(0)=1$ and $s(2 \pi)=-1$.

Solution. First

$$
v(t)=\int a(t) d t=\int \sin t+\cos t d t=-\sin t+\cos t+c
$$

Now

$$
s(t)=\int-\sin t+\cos t+c d t=-\cos t-\sin t+c t+d
$$

But $s(0)=-1+0-0+0+d=1$ so $d=2$. Then $s(\pi)=-1+0+\pi c+2=-1$ so $2 \pi c=-2 \Rightarrow c=-1 / \pi$. Thus, $s(t)=-\cos t-\sin t-\frac{t}{\pi}+2$.

### 40.3 Constant Acceleration: Gravity

In many motion problems the acceleration is constant. This happens when an object is thrown or dropped and the only acceleration is due to gravity. In such a situation we have

- $a(t)=a$, constant acceleration
- with initial velocity $v(0)=v_{0}$
- and initial position $s(0)=s_{0}$.

Then

$$
v(t)=\int a(t) d t=\int a d t=a t+c
$$

But

$$
v(0)=a \cdot 0+c=v_{0} \Rightarrow c=v_{0}
$$

So

$$
v(t)=a t+v_{0} .
$$

Next,

$$
s(t)=\int v(t) d t=\int a t+v_{0} d t=\frac{1}{2} a t^{2}+v_{0} t+c
$$

At time $t=0$,

$$
s(0)=\frac{1}{2} a(0)^{2}+v_{0}(0)+c=s_{0} \Rightarrow c=s_{0} .
$$

Therefore

$$
s(t)=\frac{1}{2} a t^{2}+v_{0} t+s_{0}
$$

EXAMPLE 40.3.1. Suppose a ball is thrown with initial velocity $29.4 \mathrm{~m} / \mathrm{s}$ from a roof top 132.3 meters high. The acceleration due to gravity is constant $a(t)=-9.8 \mathrm{~m} / \mathrm{s}^{2}$. Find $v(t)$ and $s(t)$. Then find the maximum height of the ball and the time when the ball hits the ground.

Solution. Recognizing that $v_{0}=29.4$ and $s_{0}=132.3$ and that the acceleration is constant, we may use the general formulas we just developed.

$$
v(t)=a t+v_{0}=-9.8 t+29.4
$$

and

$$
s(t)=\frac{1}{2} a t^{2}+v_{0} t+s_{0}=-4.9 t^{2}+29.4 t+132.3
$$

The max height occurs when the velocity is 0 :

$$
v(t)=-9.8 t+29.4=0 \Rightarrow t=3 \Rightarrow s(3)=-44.1+88.2+132.3=176.4
$$

The ball hits the ground when $s(t)=0$.

$$
s(t)=-4.9 t^{2}+29.4 t+132.3=-4.9\left(t^{2}-6 t-27\right)=-4.9(t-9)(t+3)=0
$$

So $t=9$ and $(t \neq-3)$.
YOU TRY IT 40.1. (British system.) A stone is thrown upward with an initial velocity of $48 \mathrm{ft} / \mathrm{s}$ from the edge of a cliff 64 ft above a river. (Remember: Using feet, acceleration due to gravity is $-32 \mathrm{ft} / \mathrm{s}^{2}$.)
(a) Find the velocity of the stone for $t \geq 0$.
(b) Find the position of the stone for $t \geq 0$.
(c) Find the time when it reaches its highest point (and the height).
(d) Find the time when the stone hits the ground.

EXAMPLE 40.3.2. A person drops a stone from a bridge. What is the height (in meters) of the bridge if the person hears the splash 5 seconds after dropping it?

Solution. Here's what we know. $v_{0}=0$ (dropped) and $s(5)=0$ (hits water). And we know acceleration is constant, $a=-9.8 \mathrm{~m} / \mathrm{s}^{2}$. We want to find the height of the bridge, which is just $s_{0}$. Use our constant acceleration motion formulas to solve for a.

$$
v(t)=a t+v_{0}=-9.8 t
$$

> Answers: $v(t)=-32 t+48, s(t)=$ $-16 t^{2}+48 t+64, \max \mathrm{ht}: 100 \mathrm{ft}$ at $t=1.5 \mathrm{~s}$, hits ground at $t=4 \mathrm{~s}$.
and

$$
s(t)=\frac{1}{2} a t^{2}+v_{0} t+s_{0}=-4.9 t^{2}+s_{0}
$$

Now we use the position we know: $s(5)=0$.

$$
s(5)=-4.9(5)^{2}+s_{0} \Rightarrow s_{0}=122.5
$$

Notice that we did not need to use the velocity function.
YOU TRY IT 40.2 (Extra Credit). In the previous problem did you take into account that sound does not travel instantaneously in your calculation above? Assume that sound travels at $340.3 \mathrm{~m} / \mathrm{s}$. What is the height (in m ) of the bridge if the person hears the splash 5 seconds after dropping it?

EXAMPLE 40.3.3. A stone dropped off a cliff hits the ground with speed of $44.1 \mathrm{~m} / \mathrm{s}$. What was the height of the cliff?

Solution. Recognizing that $v_{0}=0$ and $s_{0}$ is unknown and is the cliff height and that the acceleration is constant $a=-9.8 \mathrm{~m} / \mathrm{s}$, we may use the general formulas for motion with constant acceleration:

$$
v(t)=a t+v_{0}=-9.8 t+0=-9.8 t \text {. }
$$

Now we use the velocity and the one velocity we know $v=-44.1$ when it hits the ground so

$$
v(t)=-9.8 t=-44.1 \Rightarrow t=\frac{-44.1}{-9.8}=4.5
$$

when it hits the ground. Now remember when it hits the ground the height is 0 . So $s(9 / 2)=0$. But we know

$$
s(t)=\frac{1}{2} a t^{2}+v_{0} t+s_{0}=-4.9 t^{2}+0 t+s_{0}=-4.9 t^{2}+s_{0} .
$$

Now substitute in $t=4.5$ and solve for $s_{0}$.

$$
s(4.5)=0 \Rightarrow-9.8(4.5)^{2}+s_{0}=0 \Rightarrow s_{0}=198.45
$$

The cliff height is 198.45 m .
EXAMPLE 40.3.4. A car is traveling at $90 \mathrm{~km} / \mathrm{h}$ when the driver sees a deer 75 m ahead and slams on the brakes. What constant deceleration is required to avoid hitting Bambi? [Note: First convert $90 \mathrm{~km} / \mathrm{h}$ to $\mathrm{m} / \mathrm{s}$.]

Solution. Let's list all that we know. $v_{0}=90 \mathrm{~km} / \mathrm{h}$ or $\frac{90000}{60 \cdot 60}=25 \mathrm{~m} / \mathrm{s}$. Let $s_{0}=0$ and let time $t^{*}$ represent the time it takes to stop. Then $s\left(t^{*}\right)=75 \mathrm{~m}$. Now the car is stopped at time $t^{*}$, so we know $v\left(t^{*}\right)=0$. Finally we know acceleration is an unknown constant, $a$, which is what we want to find.

Now we use our constant acceleration motion formulas to solve for $a$.

$$
v(t)=a t+v_{0}=a t+25
$$

and

$$
s(t)=\frac{1}{2} a t^{2}+v_{0} t+s_{0}=\frac{1}{2} a t^{2}+25 t .
$$

Now we use other velocity and position we know: $v\left(t^{*}\right)=0$ and $s\left(t^{*}\right)=75$ when the car stops. So

$$
v\left(t^{*}\right)=a t^{*}+25=0 \Rightarrow t^{*}=-25 / a
$$

and

$$
s\left(t^{*}\right)=\frac{1}{2} a\left(t^{*}\right)^{2}+25 t^{*}=\frac{1}{2} a(-25 / a)^{2}+25(-25 / a)=75 .
$$

Simplify to get

$$
\frac{625 a}{2 a^{2}}-\frac{625}{a}=\frac{625}{2 a}-\frac{1350}{2 a}=-\frac{625}{2 a}=75 \Rightarrow 150 a=-625
$$

so

$$
a=-\frac{625}{150}=-\frac{25}{6} \mathrm{~m} / \mathrm{s}
$$

The time it takes to stop is

$$
t^{*}=\frac{-25}{a}=\frac{-25}{-\frac{25}{6}}=6
$$

EXAMPLE 40.3.5. Mo Green is attempting to run the 100 m dash in the Geneva Invitational Track Meet in 9.8 seconds. He wants to run in a way that his acceleration is constant, $a$, over the entire race. Determine his velocity function. ( $a$ will still appear as an unknown constant.) Determine his position function. There should be no unknown constants in your equation. What is his velocity at the end of the race? Do you think this is realistic?

Solution. We have: constant acceleration $=a \mathrm{~m} / \mathrm{s}^{2} ; v_{0}=0 \mathrm{~m} / \mathrm{s} ; s_{0}=0 \mathrm{~m}$. So

$$
v(t)=a t+v_{0}=a t
$$

and

$$
s(t)=\frac{1}{2} a t^{2}+v_{0} t+s_{0}=\frac{1}{2} a t^{2}
$$

But $s(9.8)=\frac{1}{2} a(9.8)^{2}=100$, so $a=\frac{200}{(9.8)^{2}}=2.0825 \mathrm{~m} / \mathrm{s}^{2}$. So $s(t)=\frac{1}{2} a t^{2} \approx 1.0412 t^{2}$.
Mo's velocity at the end of the race is $v(9.8)=a(9.8)=2.0825(9.8)=20.41$
$\mathrm{m} / \mathrm{s}$... not realistic.
EXAMPLE 40.3.6. One car intends to pass another on a back road. What constant acceleration is required to increase the speed of a car from $30 \mathrm{mph}(44 \mathrm{ft} / \mathrm{s})$ to $50 \mathrm{mph}\left(\frac{220}{3} \mathrm{ft} / \mathrm{s}\right)$ in 5 seconds?

Solution. Given: $a(t)=a$ constant. $v_{0}=44 \mathrm{ft} / \mathrm{s} . s_{0}=0$. And $v(5)=\frac{220}{3} \mathrm{ft} / \mathrm{s}$.
Find $a$. But $v(t)=a t+v_{0}=a t+44$. So $v(5)=5 a+44=\frac{220}{3} \Rightarrow 5 a=\frac{220}{3}-44=\frac{88}{3}$.
Thus $a=\frac{88}{15}$.

