40.3 *Constant Acceleration: Gravity*

In many motion problems the acceleration is constant. This happens when an object is thrown or dropped and the only acceleration is due to gravity. In such a situation we have

- $a(t) = a$, constant acceleration
- with initial velocity $v(0) = v_0$
- and initial position $s(0) = s_0$.

Then

$$v(t) = \int a(t)\,dt = \int a\,dt = at + c.$$

But

$$v(0) = a \cdot 0 + c = v_0 \Rightarrow c = v_0.$$

So

$$\boxed{v(t) = at + v_0.}$$

Next,

$$s(t) = \int v(t)\,dt = \int at + v_0\,dt = \tfrac{1}{2}at^2 + v_0 t + c.$$

At time $t = 0$,

$$s(0) = \tfrac{1}{2}a(0)^2 + v_0(0) + c = s_0 \Rightarrow c = s_0.$$

Therefore

$$\boxed{s(t) = \tfrac{1}{2}at^2 + v_0 t + s_0.}$$

EXAMPLE 40.3.1. Suppose a ball is thrown with initial velocity 29.4 m/s from a roof top 132.3 meters high. The acceleration due to gravity is constant $a(t) = -9.8$ m/s^2. Find $v(t)$ and $s(t)$. Then find the maximum height of the ball and the time when the ball hits the ground.

Solution. Recognizing that $v_0 = 29.4$ and $s_0 = 132.3$ and that the acceleration is constant, we may use the general formulas we just developed.

$$\boxed{v(t) = at + v_0 = -9.8t + 29.4}$$

and

$$\boxed{s(t) = \tfrac{1}{2}at^2 + v_0 t + s_0 = -4.9t^2 + 29.4t + 132.3.}$$

The max height occurs when the velocity is 0:

$$v(t) = -9.8t + 29.4 = 0 \Rightarrow t = 3 \Rightarrow s(3) = -44.1 + 88.2 + 132.3 = 176.4.$$

The ball hits the ground when $s(t) = 0$.

$$s(t) = -4.9t^2 + 29.4t + 132.3 = -4.9(t^2 - 6t - 27) = -4.9(t-9)(t+3) = 0.$$

So $t = 9$ and $(t \neq -3)$.

YOU TRY IT 40.1. (British system.) A stone is thrown upward with an initial velocity of 48ft/s from the edge of a cliff 64ft above a river. (Remember: Using feet, acceleration due to gravity is -32ft/s^2.)

(*a*) Find the velocity of the stone for $t \geq 0$.

(*b*) Find the position of the stone for $t \geq 0$.

(*c*) Find the time when it reaches its highest point (and the height).

(*d*) Find the time when the stone hits the ground.

Answers: $v(t) = -32t + 48$, $s(t) = -16t^2 + 48t + 64$, max ht: 100 ft at $t = 1.5$s, hits ground at $t = 4$s.

EXAMPLE 40.3.2. A stone dropped off a cliff hits the ground with speed of 44.1 m/s. What was the height of the cliff?

Solution. Recognizing that $v_0 = 0$ and s_0 is unknown and is the cliff height and that the acceleration is constant $a = -9.8$ m/s, we may use the general formulas for motion with constant acceleration:

$$\boxed{v(t) = at + v_0 = -9.8t + 0 = -9.8t}.$$

Now we use the velocity and the one velocity we know $v = -44.1$ when it hits the ground so

$$v(t) = -9.8t = -44.1 \Rightarrow t = \frac{-44.1}{-9.8} = 4.5$$

when it hits the ground. Now remember when it hits the ground the height is 0. So $s(9/2) = 0$. But we know

$$\boxed{s(t) = \tfrac{1}{2}at^2 + v_0t + s_0 = -4.9t^2 + 0t + s_0 = -4.9t^2 + s_0.}$$

Now substitute in $t = 4.5$ and solve for s_0.

$$s(4.5) = 0 \Rightarrow -9.8(4.5)^2 + s_0 = 0 \Rightarrow s_0 = 198.45.$$

The cliff height is 198.45 m.

EXAMPLE 40.3.3. A car is traveling at 90 km/h when the driver sees a deer 75 m ahead and slams on the brakes. What constant deceleration is required to avoid hitting Bambi? [Note: First convert 90 km/h to m/s.]

Solution. Let's list all that we know. $v_0 = 90$ km/h or $\frac{90000}{60 \cdot 60} = 25$ m/s. Let $s_0 = 0$ and let time t^* represent the time it takes to stop. Then $s(t^*) = 75$ m. Now the car is stopped at time t^*, so we know $v(t^*) = 0$. Finally we know acceleration is an unknown constant, a, which is what we want to find.

Now we use our constant acceleration motion formulas to solve for a.

$$\boxed{v(t) = at + v_0 = at + 25}$$

and

$$\boxed{s(t) = \tfrac{1}{2}at^2 + v_0t + s_0 = \tfrac{1}{2}at^2 + 25t.}$$

Now we use other velocity and position we know: $v(t^*) = 0$ and $s(t^*) = 75$ when the car stops. So

$$v(t^*) = at^* + 25 = 0 \Rightarrow t^* = -25/a$$

and

$$s(t^*) = \tfrac{1}{2}a(t^*)^2 + 25t^* = \tfrac{1}{2}a(-25/a)^2 + 25(-25/a) = 75.$$

Simplify to get

$$\frac{625a}{2a^2} - \frac{625}{a} = \frac{625}{2a} - \frac{1350}{2a} = -\frac{625}{2a} = 75 \Rightarrow 150a = -625$$

so

$$a = -\frac{625}{150} = -\frac{25}{6} \text{ m/s}.$$

The time it takes to stop is

$$t^* = \frac{-25}{a} = \frac{-25}{-\frac{25}{6}} = 6.$$

EXAMPLE 40.3.4. Mo Green is attempting to run the 100m dash in the Geneva Invitational Track Meet in 9.8 seconds. He wants to run in a way that his *acceleration* is constant, a, over the entire race. Determine his velocity function. (a will still appear as an unknown constant.) Determine his position function. There should be no unknown constants in your equation. What is his velocity at the end of the race? Do you think this is realistic?

Solution. We have: constant acceleration $= a\,\mathrm{m/s}^2$; $v_0 = 0$ m/s; $s_0 = 0$ m. So

$$v(t) = at + v_0 = at$$

and

$$s(t) = \frac{1}{2}at^2 + v_0t + s_0 = \frac{1}{2}at^2.$$

But $s(9.8) = \frac{1}{2}a(9.8)^2 = 100$, so $a = \frac{200}{(9.8)^2} = 2.0825\,\mathrm{m/s}^2$. So $s(t) = \frac{1}{2}at^2 \approx 1.0412t^2$. Mo's velocity at the end of the race is $v(9.8) = a(9.8) = 2.0825(9.8) = 20.41$ m/s... not realistic.

EXAMPLE 40.3.5. One car intends to pass another on a back road. What constant acceleration is required to increase the speed of a car from 30 mph (44 ft/s) to 50 mph ($\frac{220}{3}$ ft/s) in 5 seconds?

Solution. Given: $a(t) = a$ constant. $v_0 = 44$ ft/s. $s_0 = 0$. And $v(5) = \frac{220}{3}$ ft/s. Find a. But $v(t) = at + v_0 = at + 44$. So $v(5) = 5a + 44 = \frac{220}{3} \Rightarrow 5a = \frac{220}{3} - 44 = \frac{88}{3}$. Thus $a = \frac{88}{15}$.

Examples From Day 41

EXAMPLE 41.1.1. A traffic engineer monitors the **rate** at which cars enter the NY Thruway outside Albany. From his data he estimates that between 4 and 6 pm, the **rate** $r(t)$ at which the cars enter the thruway is $r(t) = 100 + 1.2t - .03t^2$ cars per minute, where $t = 0$ is 4:00.

(1) Find $R(t)$, the function which describes the number of cars that have entered the thruway since 4:00 pm. [Note: $R(0) = 0$.]

(2) Find the number of cars that enter the thruway between 4:00 and 5:00. (Remember t is in minutes.)

(3) Find the number of cars that enter the thruway between 4:30 and 5:30.

Solution. We know the rate (derivative) at which the cars enter the NY Thruway.

(1) To find the number of cars, we need to take the antiderivative of the rate. So

$$R(t) = \int r(t)\,dt = \int 100 + 1.2t - .03t^2\,dt = 100t + 0.6t^2 - 0.01t^3 + c.$$

But

$$R(0) = 100(0) + 0.6(0)^2 - 0.01(0)^3 + c = 0, \text{ so } c = 0.$$

So $R(t) = 100t + 0.6t^2 - 0.01t^3$.

(2) $R(t)$ is the total number of cars that have entered the thruway since 4:00, where time t is measured in minutes. The number of cars that enter the thruway between 4:00 and 5:00 is

$$R(60) = 100(60) + 0.6(60)^2 - 0.01(60)^3 = 6000.$$

(3) The number of cars that entered the thruway between 4:30 and 5:30 is

$$R(90) - R(30) = 100(90) + 0.6(90)^2 - 0.01(90)^3 - (100(30) + 0.6(30)^2 - 0.01(30)^3) = 5190.$$

EXAMPLE 41.1.2. A ball thrown down from a roof 49 meters high reaches the ground in 3 seconds. What was its initial velocity? (Recall: Acceleration is constant and equal to $a(t) = -9.8$ m/s^2.) Find $v(t)$ and $s(t)$.

Solution. Acceleration is constant. But this time v_0 is unknown while $s_0 = 49$ m and $s(3) = 0$ (it hits the ground at 3 seconds). Since the acceleration is constant, we may use the general formulas for this situation.

$$v(t) = at + v_0 = -9.8t + v_0$$

and

$$s(t) = \tfrac{1}{2}at^2 + v_0t + s_0 = -4.9t^2 + v_0t + 49.$$

But we know that

$$s(3) = -4.9(3)^2 + v_0(3) + 49 = 0$$

which means

$$3v_0 = 4.9(9) - 4.9(10) = -4.9 \Rightarrow v_0 = -4.9/3.$$

So

$$\boxed{v(t) = -9.8t - \tfrac{49}{30}}$$

and

$$\boxed{s(t) = -4.9t^2 - \tfrac{49}{30}t + 49.}$$

The initial velocity is $v_0 = -4.9/3$ m/s.

EXAMPLE 41.1.3. My Honda Accord accelerates from 0 to 88 ft/sec (60 mph) in 13 seconds.

(1) Assume that the acceleration is constant, a. Find the particular velocity function of the car.
(2) Find the position function of the car.
(3) How far does it travel in this 13 second period?

Solution. This time $v_0 = 0$ and $s_0 = 0$ since the cart starts at position 0 with velocity 0. Further, $v(13) = 88$ ft/s. The acceleration is constant, a, but unknown.

(1) So we may use the general formulas for this situation.

$$v(t) = at + v_0 = -at.$$

So

$$v(13) = a \cdot 13 = 88 \Rightarrow a = \frac{88}{13}.$$

Thus $v(t) = \frac{88}{13}t$ ft/s.
(2) Thus $s(t) = \frac{1}{2}at^2 + v_0 t + s_0 = \frac{1}{2} \cdot \frac{88}{13}t^2 = \frac{44}{13}t^2$.
(3) So the distance travelled was $s(13) = \frac{44}{13}(13)^2 = 572$ft.

YOU TRY IT 41.2. A BMW M3 brakes from 88 ft/s (which is 60 mph) to 0 at a constant rate of $a(t) = -33$ ft/s^2.

(1) Find the corresponding velocity and distance functions.
(2) How much time does it take to stop? (This is saying something about velocity.)
(3) How far does it travel during this time? (This is about distance.)

ANSWER TO YOU TRY IT 41.2. 1. $v(t) = -33t + 88$ ft/s. 2. It takes $t = \frac{8}{3}$ s to stop. 3. The distance travelled is $s(\frac{8}{3}) = 117\frac{1}{3}$ ft, where $s(t) = -\frac{33}{2}t^2 + 88t$.

EXAMPLE 41.1.4. An oil supertanker is traveling at 16 knots (a knot is 1 nautical mile per hour and a nautical mile is 6080 feet) requires 3 nautical miles to stop with the engines in full-reverse.

(1) Find the deceleration for the tanker, assuming this rate is constant.
(2) How much time passes during this process?

Solution. This time $v_0 = 16$ knots and we may set $s_0 = 0$. The acceleration is constant, a, but unknown.

(1) So we may use the general formulas for this situation.

$$v(t) = at + v_0 = at + 16$$

and

$$s(t) = \tfrac{1}{2}at^2 + v_0 t + s_0 = \tfrac{1}{2}at^2 + 16t.$$

(2) Now we can find the time t^* that it takes to stop (velocity is 0) in terms of a. We need

$$v(t^*) = at^* + 16 = 0 \Rightarrow t^* = -\frac{16}{a}.$$

We know that position at this time is $s(t^*) = 3$ nautical miles. So

$$\begin{aligned} s(t^*) = \tfrac{1}{2}a(t^*)^2 + 16t^* &= 3 \\ \tfrac{1}{2}a(-\tfrac{16}{a})^2 + 16(-\tfrac{16}{a}) &= 3 \\ \frac{128}{a} - \frac{256}{a} &= 3 \\ -\frac{128}{a} &= 3 \\ a &= -\frac{128}{3} \text{ knots/hr} \end{aligned}$$

Now that we know a, we can determine the time t^* it took to stop.

$$t^* = -\frac{16}{a} = -\frac{16}{-\frac{128}{3}} = \frac{48}{128} = \frac{3}{8}\,\text{hr} = 22.5\,\text{min}.$$

EXAMPLE 41.1.5. In the final sprint of a crew race, the challenger is rowing at a constant velocity of 12m/s. At the point where the leader is 100m from the finish and the challenger is 15m behind, the leader is rowing at 8m/s but is accelerating at 0.5m/s/s. Who wins?

Solution. There are several strategies that can be used to solve the problem. The easiest may be to determine the time t^* it takes the challenger to finish the race and then see whether the leader has already finished or not at this time. The challenger is rowing at a constant velocity of 12m/s, so it takes $t^* = \frac{115}{12} = 9.58\overline{3}$ s to finish the race.

The leader has constant acceleration of 0.5m/s^2, with $v_0 = 8$ m/s and $s_0 = 0$ m (with 100 m still to go). So

$$s(t) = \tfrac{1}{2}at^2 + v_0 t + s_0 = \tfrac{0.5}{2}t^2 + 8t = 0.25t^2 + 8t.$$

Where is this boat at time t^*?

$$s(t^*) = s(\tfrac{115}{12}) = 0.25(\tfrac{115}{12})^2 + 8(\tfrac{115}{12}) = 99.62674\,\text{m}.$$

The challenger wins in a photo finish.

YOU TRY IT 41.3. A stone was dropped off a cliff and hit the ground with speed 120 ft/s. What was the height of the cliff?

ANSWER TO YOU TRY IT 41.3. $s_0 = 225$ feet

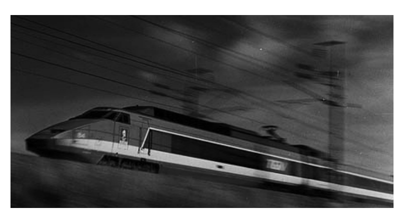
YOU TRY IT 41.4. **Extra Credit. Bring to Lab.** A sprinter in a 100m race explodes of the starting block with an acceleration of 5 m/s^2 which she sustains for the first 2 seconds. Her acceleration then drops to 0 for the remainder of the race. You will have to divide the race into two segments.

(1) Find the velocity and position function for the first 2 seconds.
(2) Find the velocity and position function for the rest of the race.
(3) What is her time for the race?

ANSWER TO YOU TRY IT 41.4. First part of race: $a = 5\,\text{m/s}^2$, $v_0 = 0$ m/s, $s_0 = 0$ m. So $v(t) = at + v_0 = 5t$ and $s(t) = \frac{1}{2}at^2 + v_0t + s_0 = \frac{5}{2}t^2$. Position at the end of 2 seconds: $s(2) = \frac{5}{2}(2)^2 = 10$ m. Velocity at the end of 2 seconds: $v(2) = 5(2) = 10$ m/sec.

(1) Second part of the race: 90 m remain at 10 m/s, so it takes her 9s. Total time: $2 + 9 = 11$s

EXAMPLE 41.1.6. France has been in the vanguard of high-speed passenger rail travel since the 1970s, and now has a modern rail network capable of accommodating trains running at speeds in excess of 84 m/s (about 300 km/h). Suppose such a train is approaching a station at 84 m/s and begins braking (say at time $t = 0$) at a constant rate of 2.8 m/s^2. How far (in meters) from the railway station did it begin to brake if it stopped right at the station platform? Use the steps below.



(1) First determine the velocity function $v(t)$.
(2) Next, determine how long it takes the train to stop.
(3) How far (in meters) from the railway station did it begin to brake if it stopped right at the station platform? (You will need the position function.)

ANSWER TO YOU TRY IT 41.1.6. We have: constant acceleration $= -2.\,\text{m/s}^2$, $v_0 = 84$ m/s; $s_0 = 0$ m.

(1) So $v(t) = at + v_0 = -2.8t + 84$.
(2) The train stops when $v(t) = -2.8t + 84 = 0$. Solving gives $t = 84/1.4 = 30$ s
(3) And $s(t) = \frac{1}{2}at^2 + v_0t + s_0 = -1.4t^2 + 84t$. So the train travels $s(30) = -1.4(30)^2 + 84(30) = 1260$ m.