The Limit Definition

We are ready to make precise what we mean the term 'limit.' So far we have worked with an intuitive understanding of the term:

DEFINITION 6.1.1 (Informal Definition of Limit). We write $\lim_{x \to a} g(x) = L$ and say that **the limit of** $g(x)$ **as** x **approaches** a if we can make $g(x)$ arbitrarily close to L by taking x sufficiently close to (but not equal to) a.

One of the points that distinguishes mathematics from other disciplines is the preciseness of its definitions. So compare the definition above to the

DEFINITION 6.1.2 (Formal Definition of Limit). Let f be a function defined on some open interval containing a, except perhaps at a itself. We say that $\lim_{x \to a} f(x) = L$ if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ so that

$$\text{if} \quad 0 < |x - a| < \delta, \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

Each inequality makes precise some aspect of the informal definition of limit.

- $|f(x) - L| < \varepsilon$ means "we can make $f(x)$ arbitrarily close to L" (within ε of L)
- by taking $|x - a| < \delta$, i.e.,"taking x sufficiently close to a" (within δ of a)
- The inequality $0 < |x - a|$ simply means "x is not equal to a."

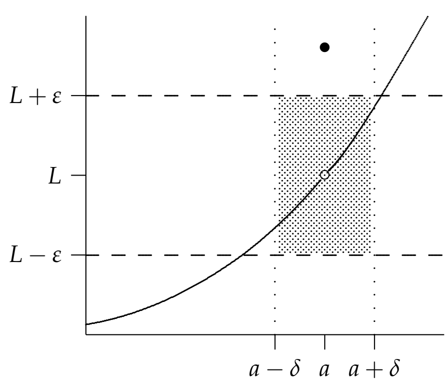
EXERCISE 6.1.3. Turn the points above into inequalities:

- $|f(x) - L| < \varepsilon$ becomes
- $|x - a| < \delta$ becomes
- $0 < |x - a|$ becomes

Examples of Using the Limit Definition

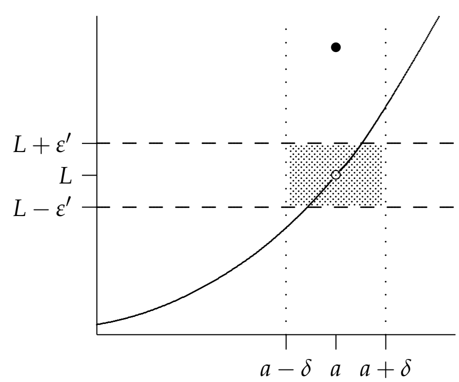
EXAMPLE 6.1.4. On the left: For the given ε, the selected δ keeps $f(x)$ within the horizontal band, that is, within ε of L over the interval from $a - \delta$ to $a + \delta$ (except perhaps at a). On the right: With a smaller ε', the old δ may fail. Is it possible to find a new δ that works?

$|y - L| < \varepsilon$
Points within
ε of L



$|x - a| < \delta$: Points within δ of a

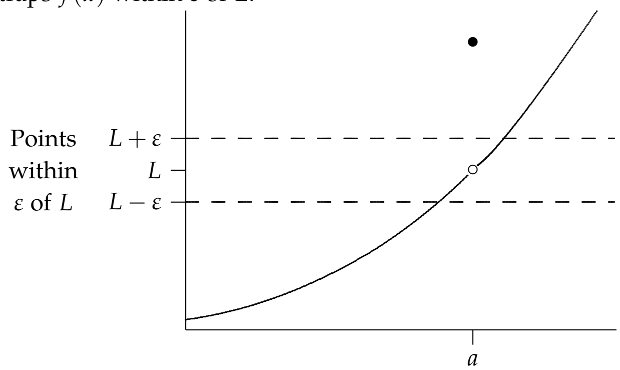
New ε'



Points within old δ of a

On the left: When x is trapped between $a - \delta$ and $a + \delta$, then $f(x)$ is trapped between $L - \varepsilon$ and $L + \varepsilon$ EXCEPT at $x = a$.

EXAMPLE 6.1.5. The formal definition says that for each and every ε, we need to be able to find a δ. With the same function as before, for a smaller choice of ε, the earlier value of δ might not work. However: As long as we can find a new δ for each new ε, the limit will exist. **For the smaller choice of ε, draw a δ interval about a that satisfies the limit definition**—that traps $f(x)$ within ε of L.

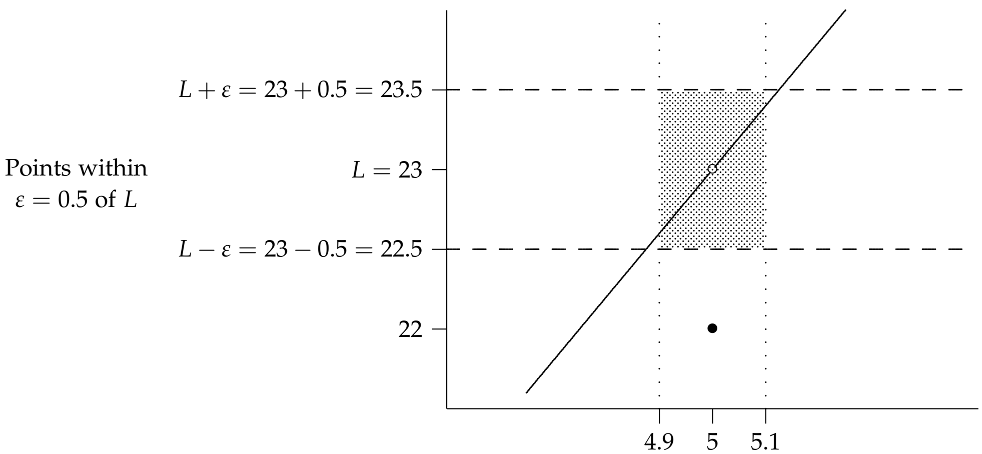


Find a δ so that if $a-\delta < x < a+\delta$ (and $x \neq a$), then $L-\varepsilon < f(x) < L+\varepsilon$. Find the vertical strip to trap $f(x)$ in the given horizontal strip.

To repeat: As long as we can find a new δ for each new ε, the limit will exist.

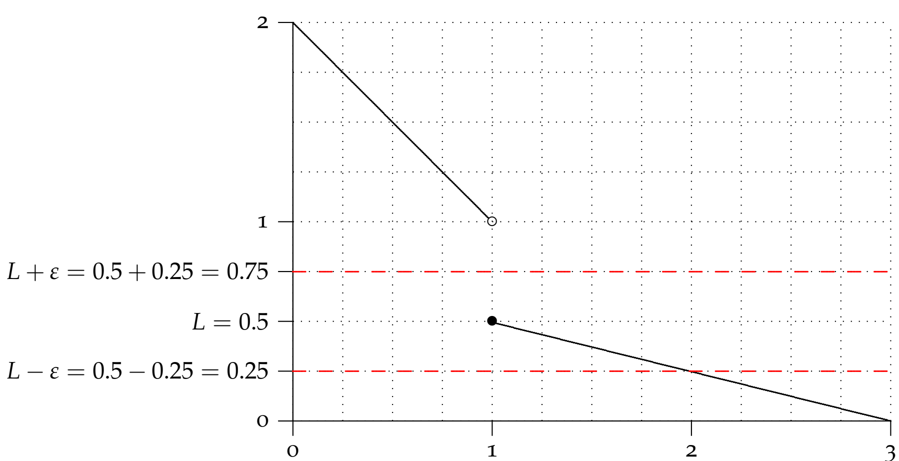
EXAMPLE 6.1.6. For the function $f(x) = \begin{cases} 4x+3, & \text{if } x \neq 5 \\ 22, & \text{if } x = 5 \end{cases}$.

Use absolute values to describe the shaded region.



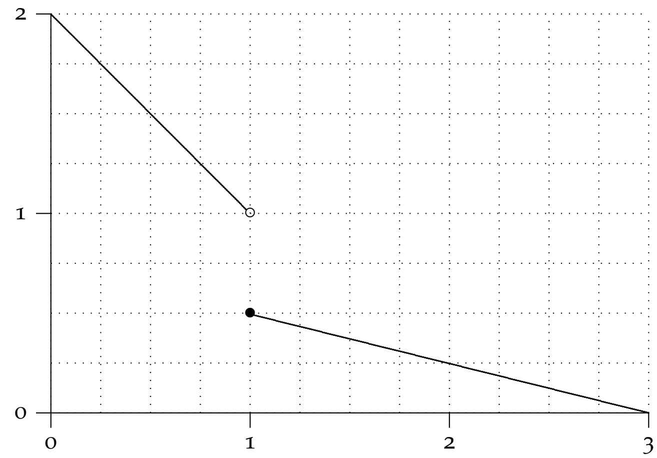
Points within .1 of 5

EXAMPLE 6.1.7. Let $f(x) = \begin{cases} 2-x, & \text{if } x < 1 \\ 1-\frac{1}{2}x, & \text{if } x \geq 1 \end{cases}$. Intuitively we can see that $\lim_{x\to 1} f(x)$ DNE. For example we show that $\lim_{x\to 1} f(x) \neq 0.5$ by using $\varepsilon = 0.25$. Now there is NO $\delta > 0$ that satisfies the limit definition. That is, for every $\delta > 0$, if $0 < |x-1| < \delta$, then $|f(x) - 0.5| > 0.25 = \varepsilon$. There are always points outside (above) the red 'ε'-corridor no matter how small δ is.

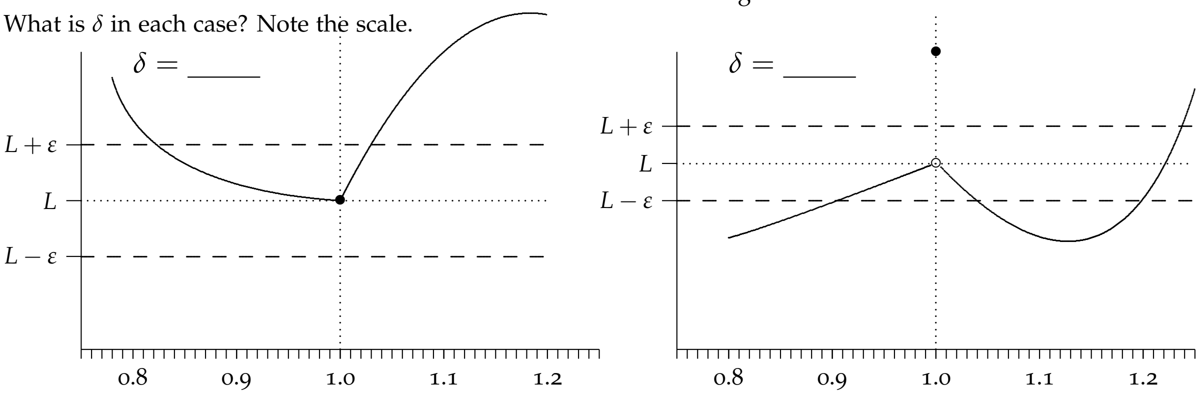


There is no δ that will trap $f(x)$ within 0.25 of 0.5. There is no δ so that if $0 < |x-1 < \delta$, then $|f(x) - 0.5| < 0.25 = \varepsilon'$.

YOU TRY IT 6.1. ☞ Here's another copy of the graph of the function in Example 6.1.7. Find a particular value of ε to show that $\lim_{x\to 1} f(x) \neq 1$.

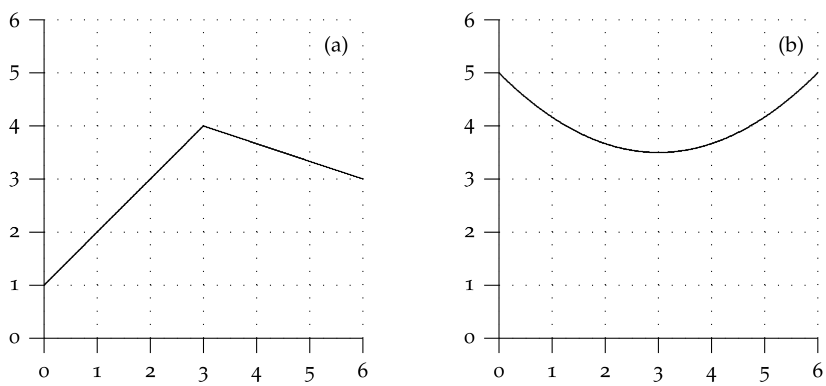


YOU TRY IT 6.2. In each figure below, for the given choice of ε, find and draw a δ interval (a vertical strip) about $a = 1$ which satisfies the limit definition—that traps the function in the horizontal band. Note: The same δ must work on both the left and right at sides of $a = 1$. What is δ in each case? Note the scale.



EXAMPLE 6.1.8. We know how to solve inequalities involving absolute values. Now apply this to functions. Determine whether the following statement is true or false for each of the functions graphed below.

$$\text{If } |x-3| < 2, \text{ then } |f(x) - 4| < 1.$$



SOLUTION. Convert the absolute values to ordinary inequalities and draw the strips.

(*a*) Do this now for $|x-3| < 2$ and $|f(x) - 4| < 1$ and draw the corresponding strips on the graphs above. Remember $f(x)$ represents the y-coordinate

(*b*) For the statement, "If $|x-3| < 2$, then $|f(x)-4| < 1$." to be true, means when x is in the vertical strip, then $f(x)$ must be in the horizontal strip. In other words, the graph of the function must be inside the rectangle when $|x-3| < 2$. Do both functions satisfy this condition?

Limit Proofs: A Two-Step Process

This section demonstrates how mathematicians use the formal definition of limit to give careful proofs of limit calculations. We will look at some very simple examples that illustrate this idea.[1] We will use a two-stage process to prove that $\lim_{x\to a} f(x) = L$. Remember that

$\lim_{x\to a} f(x) = L$ if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ so that

$$\text{if} \quad 0 < |x-a| < \delta, \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

So we need to

1. SCRAP WORK: Find δ. We do this by letting ε be an arbitrary positive number and then we use the inequality $|f(x) - L| < \varepsilon$ to work 'backwards' to a statement of the form $|x-a| < \delta$, where δ depends only on ε.

2. ARGUMENT: Write the proof. For any $\varepsilon > 0$, assume $0 < |x-a| < \delta$ and use the work in Step 1 to prove that $|f(x) - L| < \varepsilon$. (Here we work 'forward.')

The first step is essentially 'scrap work' for the proof. The second step is the actual 'proof' that our choice of δ works. This is clearer in an example.

EXAMPLE 6.1.9 (The Limit of a Linear function)**.** Prove that $\lim_{x\to 2}(3x+5) = 11$ using the formal definition of limit.

SOLUTION. SCRAP WORK: Assume that $\varepsilon > 0$ is given (but arbitrary). Find δ. In this case $a = 2$ and $L = 11$. Work backwards:

Translate from the general to this particular function.

$$|f(x) - L| < \varepsilon \overset{\text{Translate}}{\iff} |(3x+5) - 11| < \varepsilon$$

Now simplify the absolute value.

$$\overset{\text{Simplify}}{\iff} |3x - 6| < \varepsilon$$

Factor out the constant in front of x.

$$\overset{\text{Factor}}{\iff} 3|x-2| < \varepsilon$$

Solve for $|x-a|$.

$$\overset{\text{Solve}}{\iff} |x-2| < \frac{\varepsilon}{3}.$$

We now have $|x-a| < \delta$ where $a = 2$ and $\delta = \frac{\varepsilon}{3}$.

$$\therefore \text{ Choose } \delta = \tfrac{\varepsilon}{3}.$$

At this last step we have an inequality of the form $|x-a| < \delta$. We identify δ as $\frac{\varepsilon}{3}$. Notice how δ depends on ε. In particular, as ε gets smaller, so does δ. We saw this geometrically in the graphs in the first part of this handout. Now we are ready to write the actual proof.

ARGUMENT: Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{3}$. If $0 < |x-2| < \delta = \frac{\varepsilon}{3}$, then

$$|f(x) - L| = |(3x+5) - 11| \overset{\text{Simplify}}{=} |3x - 6|$$
$$\overset{\text{Factor}}{=} 3|x-2|$$

Because we know $|x-2| < \frac{\varepsilon}{3}$, we can substitute and make an inequality

$$\overset{|x-2| < \frac{\varepsilon}{3}}{<} 3 \cdot \frac{\varepsilon}{3} = \varepsilon.$$

The proof is complete: We have shown that for any $\varepsilon > 0$, if $0 < |x-2| < \delta = \frac{\varepsilon}{3}$, then $|(3x+5) - 11| < \varepsilon$. Having done the scrap work, the proof consists of three sentences.

☞ There are more examples in the online notes. Check them out!

☞ if $\varepsilon = 0.01$, what would δ be? Or if $\varepsilon = 0.0001$, how would you choose δ?

[1] To learn about more complex limit calculations, take Math 331.

We are trying to show that

if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$

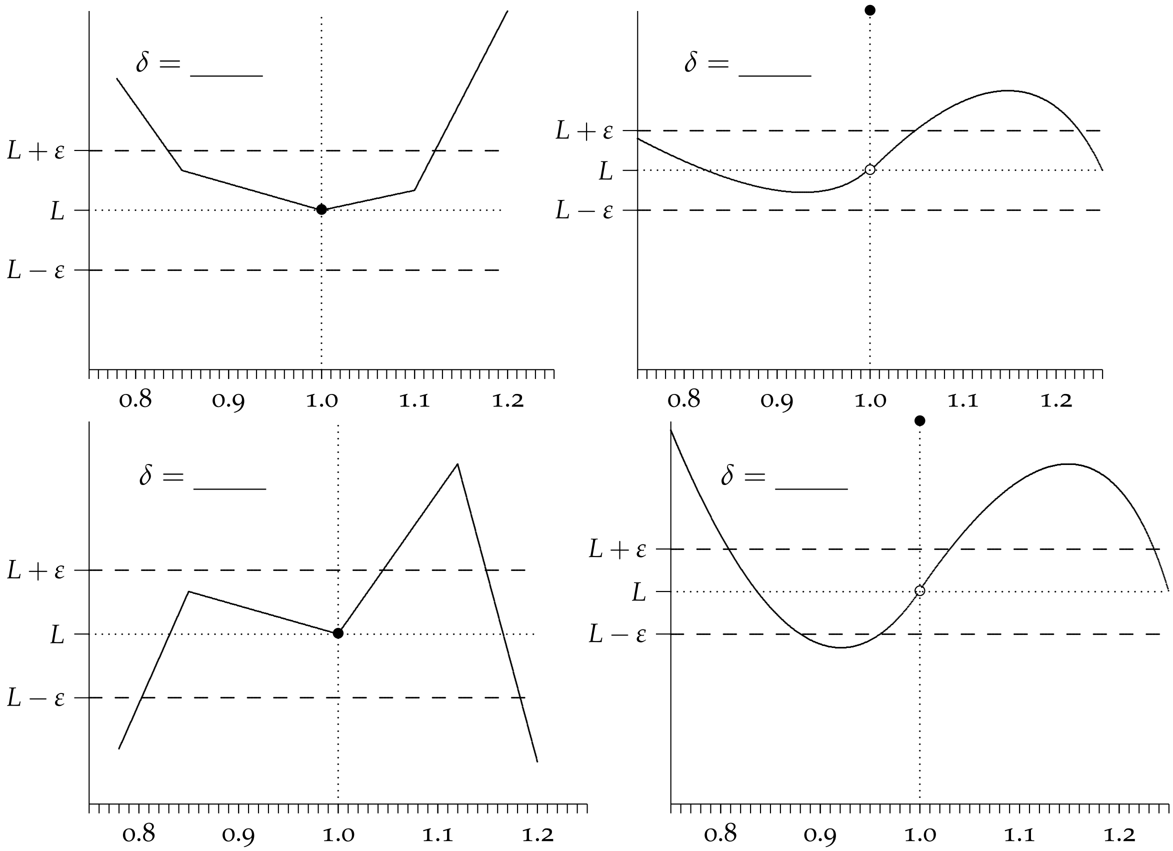
or in this particular case

if $0 < |x-2| < \frac{\varepsilon}{3}$, then $|(3x+5) - 11| < \varepsilon$.

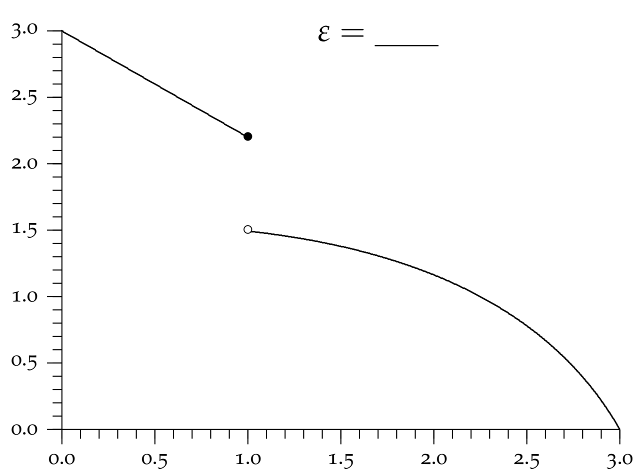
The Stuff You Need to Turn In Next Class. Name(s):

You should work with one partner if at all possible. If you do, **hand in one sheet for both of you**. Also remember: WeBWorK Set Day08 due Monday night.

1. In each figure, for the given choice of ε, find and **draw** a δ interval (a vertical strip) about $a = 1$ which satisfies the limit definition. **What is δ in each case?** Note the **scale**. **Note**: In each figure, the same δ must work on both the left and right at sides of $a = 1$.



2. For the function $f(x)$ below, show that that 1.5 is **not** $\lim_{x\to 1} f(x)$. To do this, **find and draw** a horizontal ε interval about $y = 1.5$ for which there is **no** value of δ that will satisfy the limit definition. (For your value of ε, you can never trap $f(x)$ in the corresponding horizontal band.) What is the value of your ε?



3. (*a*) With your partner use the **formal definition** of limit to prove the following:

$$\lim_{x\to 10} 6x - 7 = 53.$$

Use the same type of careful argument that we made in class today with absolute values, ε, and δ. See Example 6.1.9.

SCRAP WORK:

ARGUMENT:

(*b*) Suppose I told you that $\varepsilon = 0.06$. Use your work above to tell me what δ I should use.

(*c*) Suppose I told you that $\varepsilon = 0.0003$. Use your work above to tell me what δ I should use.