5-Minute Review: Polynomial Functions

You should be familiar with polynomials, which are among the simplest of functions.

Note: These 5-minute reviews will be a feature of class for the first few weeks. They cover material that you should already know. They are meant to alert you to material that that is essential to the course and that you should review if it is not familiar to you.

DEFINITION 0.1.1. A **polynomial** is a function of the form

$$y = p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a *non-negative integer* and each a_i is a real number (constant). If $a_n \neq 0$, then n is the **degree** of the polynomial (or highest power). The domain of a polynomial is all real numbers, $(-\infty, \infty)$.

Degree 1 polynomials: Have the form $y = a_1 x + a_0$ and are just equations of lines. The more familiar form is $y = mx + b$ (where $m = a_1$ is the slope and $b = a_0$ is the y-intercept).

Degree 2 polynomials: Have the form $y = f(x) = a_2 x^2 + a_1 x + a_0$ or $y = ax^2 + bx + c$. These are the familiar *quadratic functions* or *parabolas*.

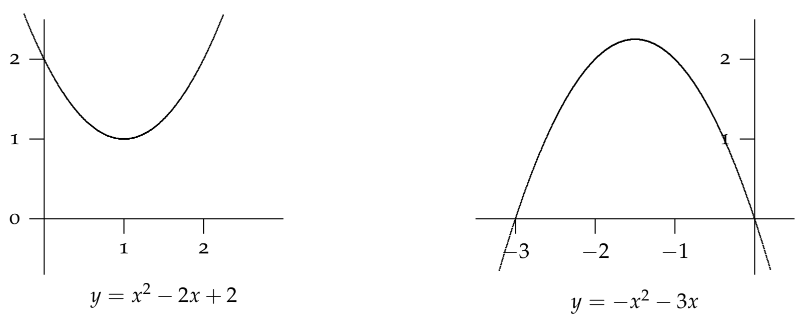

Figure 1: Two 'parabolas' (degree 2 polynomials). When the coefficient of x^2 is negative, the parabola is upside-down.

Degree 3 polynomials: Have the form $y = f(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$. Such polynomials are also called *cubics* since the degree is 3. Such polynomials can have either 1, 2, or 3 roots.

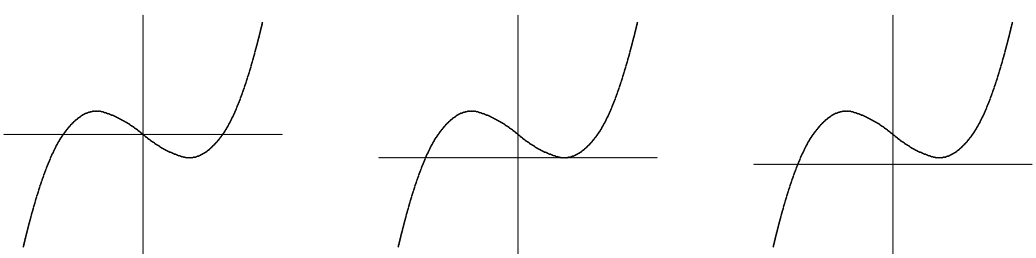

Figure 2: Three cubics (degree 3 polynomials) with 3, 2, and 1 roots, respectively.

Degree 0 polynomials: A bit sneaky. The definition of a polynomial says that all the powers must be non-negative integers. Well, 0 is a non-negative integer. So a degree 0 polynomial would have the form $y = ax^0$. Since any number x raised to the zero power is 1, then a degree 0 polynomial is really just a constant function: $y = a$. For example, the following are all degree 0 polynomials:

$$y = 8, \qquad , y = -\frac{7}{3}, \quad y = 25, \quad \text{and} \quad y = 0.$$

EXAMPLE 0.1.2. Determine which of the following are polynomials. For those that are, what are their degrees?

(*a*) $-\frac{2}{3}x^5 + 3x^4 + x^2 - 11$ (*b*) $5x^2 - x^{1/3} - 23$ (*c*) 4 (*d*) $6x^{-2} + 4x^{-1} + 2$

(*e*) $\sqrt{3x^{12} + 11x^9 + 12}$ (*f*) $\dfrac{2x^2 + x}{7x + 1}$ (*g*) $\sin(3x^2 + 1)$ (*h*) $-4x + 6^{1/2}x^3 + 7$

SOLUTION. (*a*) $-\frac{2}{3}x^5 + 3x^4 + x^2 - 11$ is a polynomial and the degree is 5.

(*b*) $5x^2 - x^{1/3} - 23$ is not a polynomial since one of the powers is not a non-negative integer.

(*c*) 0 is a polynomial. It is equivalent to $0x^0$. The degree is 0.

(*d*) $6x^{-2} + 4x^{-1} + 2$ is not a polynomial since some of the powers are not non-negative integers.

(*e*) $\sqrt{3x^{12} + 11x^9 + 12}$ is not a polynomial. The function underneath the square root is a polynomial, but the composition with the square root results in a non-polynomial function.

(*f*) $\frac{2x^2+x}{7x+1}$ is not a polynomial even though both the numerator and denominator are.

(*g*) $\sin(3x^2 + 1)$ is not a polynomial. The inner function is, but composition with the sine function results in a non-polynomial.

(*h*) $-4x + 6^{1/2}x^3 + 7$ is a polynomial and its degree is 3.

YOU TRY IT 0.1. Identify which functions are polynomials and determine their degree.

(*a*) $p(x) = -4x^3 + 2x + 11$ (*b*) $q(x) = \dfrac{1}{5x^2} - \dfrac{7}{x}$ (*c*) $r(t) = \dfrac{t^2 + 1}{t^2 - 1}$

(*d*) $p(x) = \sin(x^2 + 1)$ (*e*) $s(x) = 2x^2 - x^{1/2} + 7$ (*f*) $q(t) = \sqrt{t^3 + t^2 + 1}$

(*g*) $r(x) = 11$ (*h*) $r(x) = 3^{1/2}x^4 - 2x + \pi$ (*i*) $f(x) = -\dfrac{2}{3}x^5 + 3x^4 + x^2 - 11$

(*j*) $p(x) = 5x^2 - x^{1/3} - 23$ (*k*) $g(x) = 6x^{-2} + 4x^{-1} + 2$ (*l*) $q(x) = 3x - 4x^2 + \dfrac{x^3}{6}$

ANSWER TO YOU TRY IT 0.1. Polynomials (degree): a (3), g (0), h (4), i (5), l (3).