Day 39 — Strategy: Check Abs Conv First

#1a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n^4+1}}$. Check absolute convergence first. ↙ Show the absolute values

↙ Terms now positive.

(a-c) $\sum \left| \frac{(-1)^n}{\sqrt[3]{n^4+1}} \right| = \sum \frac{1}{\sqrt[3]{n^4+1}} \approx \sum \frac{1}{n^{4/3}}$. Use Limit (or Direct) Comparison

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt[3]{n^4+1}} \cdot \frac{n^{4/3}}{1} \overset{HP}{=} \lim_{n\to\infty} \frac{n^{4/3}}{\sqrt[3]{n^4}} = \lim_{n\to\infty} \frac{n^{4/3}}{n^{4/3}} = 1$$

Since $\sum \frac{1}{n^{4/3}}$ converges (p-series, $p = 4/3 > 1$) and since $0 < L = 1 < \infty$ By the limit comp. Test $\sum \left| \frac{(-1)^n}{\sqrt[3]{n^4+1}} \right|$ converges, so $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n^4+1}}$ <u>converges absolutely</u>.

b) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n+10}}$. Check Abs. Conv. $\sum_{n=1}^{\infty} \left| \frac{\cos(n\pi)}{\sqrt{n+10}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+10}} \approx \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

(a-b) use Limit Comp w/ $\sum \frac{1}{n^{1/2}}$. The terms of both are positive

(d) $L = \lim_{n\to\infty} \frac{1}{\sqrt{n+10}} \cdot \frac{n^{1/2}}{1} \overset{HP}{=} \lim_{n\to\infty} \frac{n^{1/2}}{n^{1/2}} = 1$. (e) Since $0 < L = 1 < \infty$

and since $\sum \frac{1}{n^{1/2}}$ <u>diverges</u> (p-series, $p = 1/2 \le 1$), then by Limit comparison $\sum \left| \frac{\cos(n\pi)}{\sqrt{n+10}} \right|$ diverges. (Not Abs Conv).

Must check Conditional Convergence w/ Alternating Series Test $a_n = \frac{1}{\sqrt{n+10}}$, Check the two conditions:

(1) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+10}} = 0$ ✓

(2) Decreasing? Notice $\frac{1}{\sqrt{(n+1)+10}} < \frac{1}{\sqrt{n+10}}$ so $a_{n+1} \le a_n$

(or use $f'(x) = -\frac{1}{2}(x+10)^{-3/2} < 0$ for x in $[1, \infty)$)

By the Alt. Ser. Test $\sum \frac{\cos(n\pi)}{\sqrt{n+10}}$ converges. So Overall it $\boxed{\text{converges conditionally}}$.

\# 1(c) $\sum_{k=1}^{\infty} \frac{(-7)^{k+1}}{k!}$. Perfect for Ratio Test Extension. Take

$$r = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-7)^{k+2}}{(k+1)!}\cdot\frac{k!}{(-7)^{k+1}}\right| = \lim_{k\to\infty}\left|\frac{-7}{k}\right| = 0 < 1$$

Since $r<1$, By the Ratio Test extension, $\sum_{k=1}^{\infty}\frac{(-7)^{k+1}}{k!}$ <u>converges absolutely</u>.

\# 2 a) $\sum_{k=2}^{\infty}\frac{(-1)^k}{\ln k}$. Check Abs Conv first. $\sum_{k=2}^{\infty}\left|\frac{(-1)^k}{\ln k}\right| = \sum_{k=2}^{\infty}\frac{1}{\ln k}$. Use limit comparison w/ $\sum\frac{1}{k}$. Both series have positive terms

$$L = \lim_{n\to\infty}\frac{1}{\ln k}\cdot\frac{k}{1} = \lim_{x\to\infty}\frac{x}{\ln x} \overset{LH}{=} \lim_{x\to\infty}\frac{1}{\frac{1}{x}} = \lim_{x\to\infty}\frac{x}{1} = \infty$$

Since $L=\infty$ and since $\sum\frac{1}{k}$ diverges (p-series, $p=1$), $\sum_{k=2}^{\infty}\left|\frac{(-1)^k}{\ln k}\right|$ diverges

Must check conditional convergence w/ Alt. Series test
$a_k = \frac{1}{\ln k}$ is positive and check the 2 conditions

(1) $\lim_{k\to\infty} a_k = \lim_{k\to\infty}\frac{1}{\ln(k)} = 0$ ✓

(2) Decreasing? Use $\ln(k+1) > \ln k$ so $\frac{1}{\ln(k+1)} < \frac{1}{\ln k}$ (or take the derivative: $f'(x) = \frac{-1}{x(\ln x)^2} < 0$ for x in $[1,\infty)$)

So a_n is decreasing. By the alternating series test $\sum_{k=2}^{\infty}\frac{(-1)^k}{\ln k}$ converges. Overall: Converges Conditionally

2 (b) $\sum_{n=1}^{\infty}(-1)^n\frac{(n!)^3}{(3n)!}$. Use the ratio test extension (factorials)

$$r = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}[(n+1)!]^3}{(3(n+1))!}\cdot\frac{(3n)!}{(-1)^n(n!)^3}\right| = \lim_{n\to\infty}\left|\frac{(-1)(n+1)^3(3n)!}{(3n+3)!}\right|$$

$$= \lim_{n\to\infty}\frac{(n+1)^3}{(3n+1)(3n+2)(3n+3)} \overset{HP}{=} \lim_{n\to\infty}\frac{n^3}{27n^3} = \frac{1}{27} < 1$$

Since $r<1$, by the ratio test extension, the series <u>converges absolutely</u>