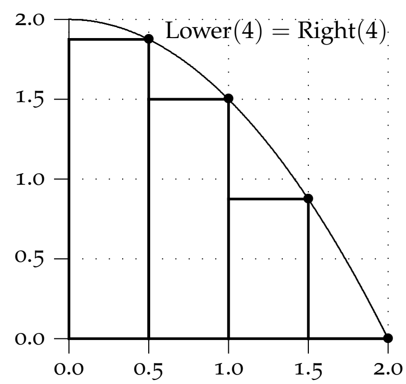
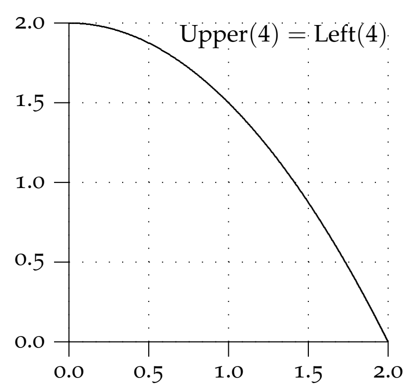
My Office Hours: M & W 2:30–4:00, Tu 2:00–3:30, & F 1:30–2:30 or by appointment. **Math Intern:** Sun: 2:00–5:00, 7:00–10pm; Mon thru Thu: 3:00–5:30 and 7:00–10:30pm in Lansing 310. Website: `http://math.hws.edu/~mitchell/Math131F15/index.html`.

☕ *Review*

Review the online notes for today's lecture. In Sections 5.1–5.2, your text concentrates on right, left, and midpoint Riemann sums, which we will denote Right(n), Left(n), and Midpoint(n). In class today we will discuss upper and lower sums, Upper(n) and Lower(n). When it actually comes to computing one of these, it will often turn out to be a right-endpoint or left-endpoint Riemann sum because the functions are always increasing or decreasing. *There are two aspects of Riemann sums that are important to understand. There is all the* ***symbolism*** *and summation machinery and there is the underlying* ***geometry*** *of what such a sum represents. The material will only make sense if you understand both aspects.* The first two problems illustrate this.

☕ *Model Examples*





1. **Geometry**. Let $f(x)$ be the function graphed below in both panels. In the margin figure, I have drawn the Riemann sum Lower(4) which uses 4 equal width rectangles and the lowest points in each interval, which are at right-hand endpoints (marked by • in the diagram), to determine the heights. We can use the drawing to **estimate** the value of Lower(4) by figuring out the areas of the the rectangles. The base width of each is $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$. Now for the heights. Notice that one of the heights in this example is 0! The sum of the areas of the 4 rectangles (using height × base) are:

$$\text{Lower}(4) \approx (1.9)(0.5) + (1.5)(0.5) + (0.9)(0.5) + (0)(0.5) = 2.15$$

Notice we did not need to use any summation notation because there were so few rectangles. We can just work it out by hand. This is the **meaning** of the lower Riemann sum Lower(4). This is the same as the right-endpoint sum Right(4).

(*a*) In the margin figure, **draw** and **estimate** the Riemann sum Upper(4).

2. **Model Symbolism Problem**: The function above is actually $f(x) = 2 - \frac{1}{2}x^2$ on $[0, 2]$. Compute Lower(n) for this situation. Notice we will have to use the **right-hand** endpoints of the intervals because the function is decreasing.

(*a*) What is the width Δx of each rectangle? Ans: $\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$.

(*b*) What is the evaluation point x_k? Ans: $x_k = a + k\Delta x = 0 + k\frac{2}{n} = \frac{2k}{n}$.

(*c*) Now what is $f(x_k)$? Answer: $f\left(\frac{2k}{n}\right) = 2 - \frac{1}{2}\left(\frac{2k}{n}\right)^2 = 2 - \frac{2k^2}{n^2}$.

(*d*) OK, write out the sum Lower(n). Ans: $\text{Lower}(n) = \sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n}\left[2 - \frac{2k^2}{n^2}\right]\frac{2}{n}$. ← Remember Lower($n$) = Right($n$).

(*e*) Now use your summation formulas to simplify Lower(n). Split into 2 pieces:

$$\text{Lower}(n) = \text{Right}(n) = \frac{2}{n}\sum_{k=1}^{n} 2 - \frac{2}{n}\sum_{k=1}^{n}\frac{2k^2}{n^2} = \frac{2}{n}[2n] - \frac{4}{n^3}\sum_{k=1}^{n} k^2 = 4 - \frac{4}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] = 4 - \frac{2(n+1)(2n+1)}{3n^2}.$$

Simplifying a bit further, we get:

$$\text{Lower}(n) = \text{Right}(n) = 4 - \frac{2(2n^2+3n+1)}{3n^2} = 4 - 2\left(\frac{2}{3} + \frac{1}{n} + \frac{1}{3n^2}\right) = \frac{8}{3} - \frac{2}{n} - \frac{2}{3n^2}.$$

(*f*) To get the exact "area" evaluate $\lim_{n\to\infty} \text{Lower}(n) = \lim_{n\to\infty} \left(\frac{8}{3} - \frac{2}{n} - \frac{2}{3n^2}\right) = \frac{8}{3}$.

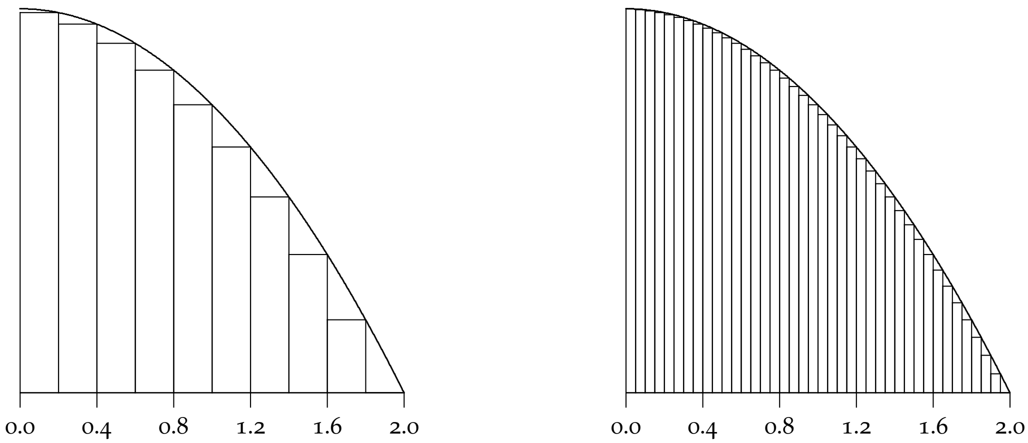


Figure 4.1: The left-hand graph shows the function $f(x) = 2 - \frac{1}{2}x^2$ on $[0, 2]$ with the interval divided in to $n = 10$ subintervals of width $\frac{2}{n} = \frac{2}{10}$. The heights of the rectangles come from the function values $f\left(\frac{2i}{n}\right) = f\left(\frac{2i}{10}\right)$ as i changes from 1 to 10. he sum of the areas of the 10 rectangles is 2.46 and is an approximation of the area under the curve on the interval $[0, 2]$.

In the right-hand graph, $n = 40$, and the same process is repeated. The sum of the areas of all the rectangles is a better approximation of the true area under the curve on the interval $[0, 2]$.

By letting $n \to \infty$, we may obtain the exact area under the curve. We found that this area was $\frac{8}{3}$ in part (f) above.

Reading, Practice, and Extra Credit

1. (*a*) Reread 5.1 and Start 5.2.

(*b*) Practice (geometry): Page 343: #9, 13, 17.

(*c*) We will almost never use midpoint Riemann sums (you will see why in another day or so). Still you should try one. Do page 344 #27. Remember: Use the function $f(x) = 100 - x^2$ to obtain the actual height at the midpoint of each interval.

2. We don't even need the graph of the function to calculate a Riemann sum. All we need are the intervals and a function value in each interval for the height. Try page 344 #35. Hint: It may be easier if you make a sketch of the function so that you can envision the left and right endpoints.

3. (*a*) **Symbolism** and manipulating sums. Page 345: #55 and 57. Use summation formulas to evaluate the sums.

(*b*) #54 (Similar to a WeBWorK problem.)

4. **Geometry**: Page 346: #65 and 67.

5. **Looking ahead.** After reading Section 5.2, try to figure out page 360 #47 or #51.

6. ⌨ **Extra Credit.** In Section 5.1, the authors use technology to evaluate Riemann sums. Since everyone has a different calculator, we won't emphasize that. However, there's a nice applet called `xFunctions` developed by the Math Department at HWS that we can use. There is a link to it on the course website or use this address: `http://math.hws.edu/~mitchell/Math131F15/Math131/xFunctions.html`. Using this software, try:

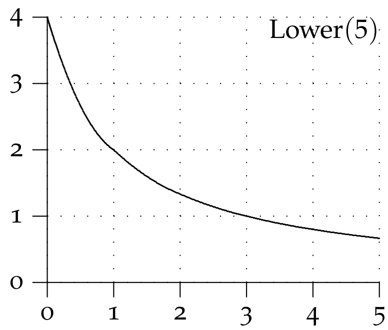
(*a*) Page 345 #54. Enter $f(x)$ as `sqrt(1 - x^2)`. Change the x and y values as needed and type in the function. Try using more intervals (there is a maximum of $n = 512$) and different types of sums (play with the buttons!)

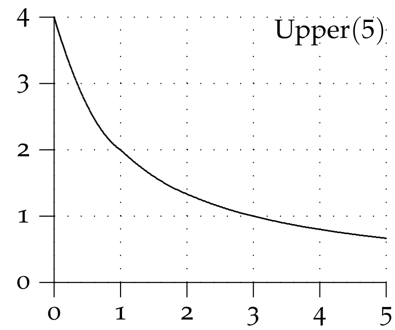
(*b*) Page 345 #46. No need to write out the sum, just do the approximations. Enter $\pi/4$ as `0.785398`.

Hand In just this sheet. Keep the other. Use the models on the Day 3 Handout to help with these.

0. Do the Day03 problems on WeBWorK. Remember to finish those assigned earlier.

1. Geometry Basics. Draw Lower(5) and Upper(5) for the function below on the interval $[0,5]$.

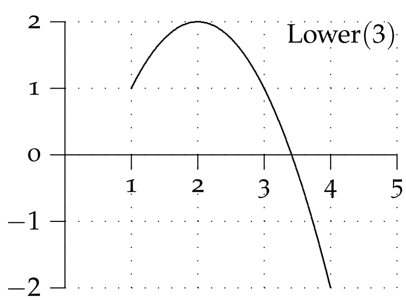


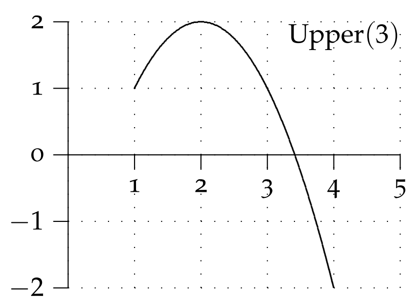


(*a*) Use the graph to estimate Lower(5). Show the terms you added.

(*b*) Use the graph to estimate Upper(5). Show the terms you added.

2. Geometry Basics. Draw Lower(3) and Upper(3) for the function below on the interval $[1,4]$. **Caution:** Watch out for your heights!





(*a*) Use the graph to estimate Lower(3). Show the terms you added.

(*b*) Use the graph to estimate Upper(3). Show the terms you added.

3. Each of the functions below is increasing so the right hand endpoints are used to find the upper sums. Fill in the following table (but do not simplify the sum):

See similar problems on WeBWorK set Day03.

$f(x)$	$[a,b]$	$\Delta x = \frac{b-a}{n}$	$x_i = a + i\Delta x$	Simplified $f(x_i)$	$\text{Upper}(n) = \sum_{i=1}^{n} f(x_i)\Delta x$
$2x^2 - 2$	$[1,3]$				$\sum_{i=1}^{n}$
$(x+1)^3$	$[-1,3]$				$\sum_{i=1}^{n}$

Over

4. (*a*) Page 343–344: #18. Use the actual function values; show the terms in each sum.

(*b*) Page 344: #36. Use the actual function values; show the 8 terms in each sum.

(*c*) Page 346 #64 Use the function values; show the 3 terms in each sum.

(*d*) Page 346: #68(a,b) Use the actual area; be careful of the scale! What units should you use in your answer?