Accumulation Functions

YOU TRY IT 2.16. Define $A(x) = \int_{-2}^{x} f(t)\,dt$, Then remember $A(x)$ is just the net area between the graph of f and the x-axis on the interval from -2 to endpoint x. Answer the following questions. (See Figure 2.44.)

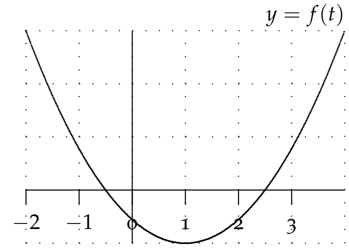

Figure 2.44: Define $A(x) = \int_{-2}^{x} f(t)\,dt$.

(1) Determine the following values of $A(x)$:

x	-2	-1	0	1	2	3	4
$A(x)$							

(2) On what interval(s) is A decreasing? Increasing? Explain.

(3) At what point(s), if any, does A have a local max? Min?

(4) Sketch a rough graph of $A(x)$ on $[0, 4]$.

2.4 *An Application of Definite Integrals: Average Value*

Here is another simple example of an application of the definite integral which points out the power of the definition of the integral as a Riemann sum.

Suppose that we want to know the average temperature for February 27, 2015 in Geneva (see Figure 2.45). How might we find it? Well, we could take the $n = 24$ hourly temperature recordings, add them together, and then divide by 24 might as we might do to find any average. Is the average 19.7 as listed in the table? What 'average' is that?

Time	Temp
1:00	12
2:00	13
3:00	13
4:00	12
5:00	11
6:00	12
7:00	13
8:00	18
9:00	21
10:00	24
11:00	26
12:00	28
13:00	28
14:00	29
15:00	29
16:00	27
17:00	25
18:00	24
19:00	21
20:00	19
21:00	17
22:00	18
23:00	17
24:00	16
Ave	19.7

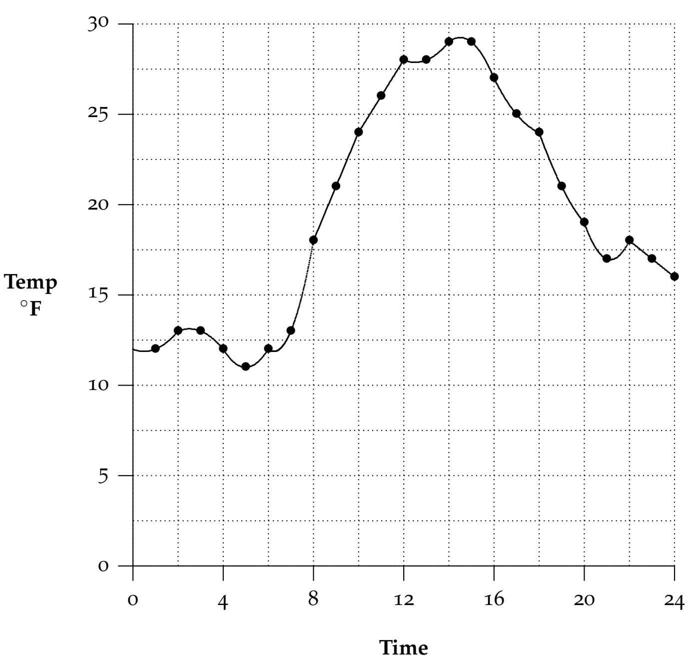

Figure 2.45: A graph of the temperature on February 27, 2003 using the data to the right.

The average of 19.7 'privileges' those recordings made on the hour. We could get a better estimate if we recorded temperatures every half-hour, or every 5 minutes, or every minute, or perhaps every second. The more recordings we use, the better the 'average.' Let's generalize the problem. In doing so, we are sometimes able to see the pattern which will help us solve the particular problem we are interested in.

The Average Value Problem: Let f be a continuous function on the closed interval $[a, b]$. Find the average value of f on $[a, b]$.

Solution. We make use of the outline of steps on page 26. But how do we subdivide an average and make it product? As usual, start by dividing $[a, b]$ into n equal subintervals with partition points $\{x_0, x_1, \ldots, x_n\}$. Then, as we suggested above,

$$\text{Average of } f \approx \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} = \sum_{k=1}^{n} f(x_k) \cdot \frac{1}{n}. \tag{2.10}$$

The summation looks almost like a Riemann sum except we now have $\frac{1}{n}$ instead of Δx. But hold on!

$$\Delta x = \frac{b-a}{n}$$

so

$$\frac{1}{n} = \frac{b-a}{n} \cdot \frac{1}{b-a} = \frac{\Delta x}{b-a}.$$

Substituting this back in equation (2.10) gives

$$\text{Average of } f \approx \sum_{k=1}^{n} f(x_k) \cdot \frac{\Delta x}{b-a} = \frac{1}{b-a} \sum_{k=1}^{n} f(x_k) \Delta x. \tag{2.11}$$

Now we do have a Riemann sum in (2.11). We have already remarked that if we let n increase (take more points in our average), we should get a more accurate approximation. The best approximation occurs when we take a limit as the number of evaluation points $n \to \infty$. In other words

$$\text{Average of } f = \lim_{n \to \infty} \frac{1}{b-a} \sum_{k=1}^{n} f(x_k) \Delta x = \frac{1}{b-a} \int_a^b f(x)\, dx. \tag{2.12}$$

We know this limit exists and equals the definite integral because f is continuous (see Theorem 1.4.2). Having carried out the steps on page 26, we are led to make the following definition.

DEFINITION 2.4.1 (Average Value). Assume that f is integrable on $[a, b]$. Then the **average value of f on $[a, b]$** is denoted by $\overline{f} = f_{\text{ave}}$ and is defined by

$$\overline{f} = f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

EXAMPLE 2.4.2. Find the average value of $f(x) = \sqrt{x}$ on $[0, 9]$. Then find a point c between 0 and 9 where $f(c) = \overline{f} = f_{\text{ave}}$.