☞ The Math Intern be on break and start regular hours on Tuesday afternoon.

Practice

Read Section 6.7 on Physical Applications (pages 460 through Example 4 on page 465.) We will only cover work (lifting) problems. Review Arc Length Section 6.5. Skip to Section 7.1; read about **integration by parts** which reverses the product rule.

1. Show that the exact arc length of $y = f(x) = x$ on $[0,1]$ is $\sqrt{2}$.

2. Show that the arc length of $y = 2 - 3x$ on $[-2,1]$ is $3\sqrt{10}$. Since this curve is a straight line segment, check your answer by using the distance formula!

Hand In Next Class

0. A Useful Fact. This idea is used over and over again in arc length integrals, including #2–4 below. Suppose that a and n are a non-zero real numbers. Show by working out each side separately that

$$1 + \left(ax^n - \frac{1}{4a}x^{-n}\right)^2 = \left(ax^n + \frac{1}{4a}x^{-n}\right)^2.$$

(By the way, notice that when $a = \frac{1}{2}$, then $a = \frac{1}{4a}$, too!

Potential Answers: $5\sqrt{10}$, 30, $3\sqrt{10}$, $\frac{e^2+1}{2}$, $\ln(1+\sqrt{2})$, 313/12, $\frac{e^2-1}{4}$, 5/4, $3\sqrt{4}$, $\frac{e^2+1}{4}$, 469/18, 3/4.

1. Find the arc length between the points $(2,4)$ and $(5,13)$ using integration. See class notes Example 7.5 in the online notes on Arc Length.

2. (*a*) Set up the arc length integral for $f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln x$ on the interval $[1,e]$.

(*b*) Evaluate your integral in part (a). Hint: See Example 2 on page 416 to see a similar simplification of the integrand. (This also a WeBWorK problem.)

3. Find the arc length of $f(x) = x^3 + \frac{1}{12}x^{-1}$ on $[1,3]$.

4. In this problem you will be a mathematician. The **hyperbolic sine** and **hyperbolic cosine** functions are defined by

$$\sinh x = \tfrac{1}{2}e^x - \tfrac{1}{2}e^{-x} \qquad \text{and} \qquad \cosh x = \tfrac{1}{2}e^x + \tfrac{1}{2}e^{-x}$$

(*a*) Using the definition of $\cosh x$, show that $\int \cosh x\,dx = \sinh x + c$.

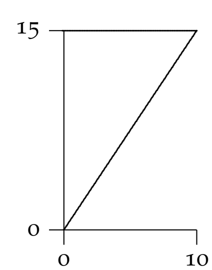
(*b*) Show that $\dfrac{d}{dx}(\cosh x) = \sinh x$. Neat!

(*c*) Show that $1 + (\sinh x)^2 = (\cosh x)^2$.

(*d*) Use your work in the first parts to determine the arc length of $\cosh x$ on $[0, \ln 2]$.

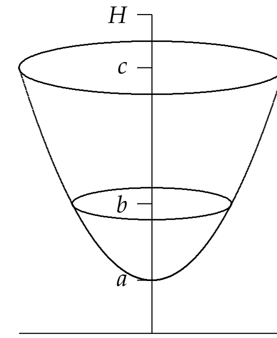
5. (*a*) (If we get this far). A cone-shaped reservoir has a 10 foot radius across the top and a 15 foot depth. If the reservoir has 9 feet of oil (density 54 lbs/ft^3) in it, how much work is required to empty it by bringing the water to the top of the reservoir? (Hint: First determine the equation of the line that determines the cone.)

(*b*) Same question with the reservoir being completely full.



Work Problems for Class Today and Next Time

Work Formula for Emptying a Tank. Assume the cross-sectional area $A(y)$ of a tank is a continuous function of the height y and that the density of the contents is a constant D. If the contents of the tank to be moved lie in the interval $[a,b]$, then the **work** done to move this material to a height H is



A tank containing a liquid between levels a and b. The top of the tank is at height c. The liquid is to be moved to a height H above the tank.

$$\text{Work} = D \int_a^b A(y)[H - y]\, dy.$$

Caution: The tank may not be full, the contents may be moved to a height H above the tank, or the entire tank may not be emptied. If the tank is being *filled* from a source at height H (either at the bottom of or below the tank), then the contents must be moved to each layer height y between a and b so the distance moved is $y - H$ rather than $H - y$.

1. (*a*) A cup shaped tank is obtained by rotating the curve $y = x^2$ about the y-axis where $0 \le x \le 3$. Assume the tank is full of 'heavy' water (density 65 lbs/ft^3). How much work is done in emptying the tank by removing the water over the top edge of the tank? (Ans: 7897.5π ft. lbs.)

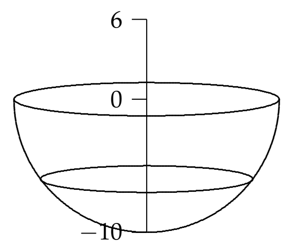
(*b*) How much work would be done in raising the water 3 feet above the tank's top? (Ans: $15,795\pi$ ft. lbs.)

(*c*) Suppose the tank is empty and is **filled** from a hole in the bottom to a depth of 3 feet. Find the work done. (Ans: 450π ft-lbs.)

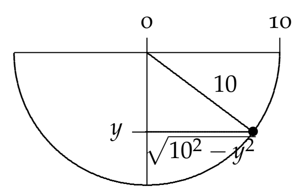
2. (*a*) A cup shaped tank is obtained by rotating the curve $y = x^3$ about the y-axis where $0 \le x \le 2$. Assume the tank is full of water (density 62.5 lbs/ft^3?). How much work is done in emptying the tank by removing the water over the top edge of the tank? (Ans: 3600π ft-lbs?)

(*b*) How much work would be done in raising the water 2 feet above the tank's top? (Ans: 6000π ft-lbs?)

(*c*) Suppose the depth of the liquid in the tank is 1 foot. Find the work required to pump the liquid to the top edge of the tank.



3. (A more complicated problem) An underground hemispherical tank with radius 10 ft is filled with oil of density 50 lbs/ft^3. Find the work done pumping the oil to the surface if the top of the tank is 6 feet below ground. It will be easiest to set up the equation of the hemisphere if we think of the top of the tank at height 0 and then pump the oil to a height of 6 feet. The cross-sections are circles. We will be able to determine the cross-sectional area once we determine the radius of the cross-section. The semi-circle is part of the circle of radius 10 centered at the origin which has equation $x^2 + y^2 = 10$. The radius of a cross-section is the x-coordinate of the point (x, y) that lies on the semi-circle in fourth quadrant. So,



$$r = x = \sqrt{(10)^2 - y^2}.$$

Therefore the cross-sectional area is

$$A(y) = \pi r^2 = \pi[10^2 - y^2] = \pi(100 - y^2).$$

Remember the liquid is pumped to height $H = 6$ in our re-casting of the problem.

$$\text{Work} = D \int_a^b A(y)[H - y]\, dy = 50 \int_{-10}^{0} \pi((10)^2 - y^2)[6 - y]\, dy = 50 \int_{-10}^{0} \pi \left(600 - 100y - 6y^2 + y^3\right) dy$$

$$= 50\pi \left(600y - 50y^2 - 2y^3 + \frac{y^4}{4}\right)\Bigg|_{-10}^{0} = 50\pi\,[0 - (-6000 - 5000 + 2000 + 2500)] = 325,000\pi \text{ ft} - \text{lbs}.$$

4. Set up the new integral for each modification of the example above and determine the work required.

(*a*) How would the integral and work change if the tank were only 5 feet below ground? (Answer: $\frac{875000}{3}\pi$ ft − lbs.)

(*b*) How would the integral and work change if the top of the tank were at ground level? (Answer: $125,000\pi$ ft − lbs.)