My Office Hours: M & W 2:30–4:00, Tu 2:00–3:30, & F 1:30–2:30 or by appointment. **Math Intern:** Sun: 2:00–5:00, 7:00–10pm; Mon thru Thu: 3:00–5:30 and 7:00–10:30pm in Lansing 310. Website: `http://math.hws.edu/~mitchell/Math131F15/index.html`.

☕ *Practice*

Read **8.4** about the integral test. **Review all of 8.3** about series. Read the online notes.

1. **Vocabulary**: Make sure you know what each of the following terms means: series, partial sum, convergent (divergent) series, geometric series, nth term test for divergence, integral test, p-series.
2. Try page 567 #9, 15, 17, 19, 25, 27, 31, 33(use a substitution!), and 47.

Four Tests

3. **The Geometric Series Test.**

   (*a*) If $|r| < 1$, then the geometric series $\sum_{n=0}^{\infty} ar^n$ converges to $\frac{a}{1-r}$.

   (*b*) If $|r| \geq 1$, then the geometric series $\sum_{n=0}^{\infty} ar^n$ diverges.

4. **The nth term test for Divergence.** If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges. (If $\lim_{n\to\infty} a_n = 0$, this test is useless.)

5. **The Integral Test.** If $f(x)$ is a **positive**, **continuous**, and **decreasing** for $x \geq 1$ and $f(n) = a_n$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\,dx$$

   either both converge or both diverge.

6. **The p-series Test.** The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1. \end{cases}$

Hand in

Finish WeBWorK Day32 and start Day33. **The Day33 problems are EXCELLENT, especially on the integral test.**

1. Here are several series. Which of them can you say diverge by the nth term test for **divergence**? For which series is this test inconclusive? Explain. **Use appropriate mathematical language**. Pretend this is a test.

   (*a*) $\sum_{n=1}^{\infty} \frac{3n+1}{2n+5}$ (*b*) $\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$ (*c*) $\sum_{n=1}^{\infty} (1.1)^n$ (*d*) $\sum_{n=1}^{\infty} \left(1+\frac{4}{n}\right)^n$

   (*e*) $\sum_{n=1}^{\infty} \sqrt[n]{n}$ (*f*) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

2. The integral test could be used to determine whether each of the following series converges or diverges. However, using the integral test is often a lot of work. For three of the series below it is possible to use one of the other tests (geometric series test or p-series test, there are some of each). Determine whether each series converges using the simplest method. Your answer should consist of a little 'argument' (a sentence or two) and any necessary calculations to show whether the series converges or diverges. (See the model arguments below. I am looking for your reasoning.) **Use appropriate mathematical language**. Parts (d), (e), and (g) are WeBWorK 33B problems.

$$(a)\ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \qquad (b)\ \sum_{n=1}^{\infty} \frac{1}{e^{2n}} \qquad (c)\ \sum_{n=1}^{\infty} \frac{1}{n^3} \qquad (d)\ \sum_{n=1}^{\infty} \frac{1}{9+n^2} \qquad (e)\ \sum_{n=1}^{\infty} \frac{3}{n^2+7n+10}$$

Extra Credit for Extra Practice. Same instructions as above. Determine whether each converges.

$$(f)\ \sum_{n=1}^{\infty} \frac{3}{\sqrt{n^3}} \qquad (g)\ \sum_{n=1}^{\infty} \frac{n}{2n^2+1} \qquad (h)\ \sum_{n=1}^{\infty} \frac{2}{n^{1.00001}} \qquad (i)\ \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$$

☞ **Remember:** Name the test you are using, show how it applies, and clearly state what the conclusion is from the test.

EXAMPLE 0.0.1. Determine whether $\sum_{n=1}^{\infty} \frac{3}{n^2+3n}$ converges.

Solution. Apply the integral test. The function $f(x) = \frac{3}{x^2+3x}$ is continuous and positive on $[1, \infty)$. It is decreasing since $f'(x) = \frac{-3(2x+3)}{(x^2+3x)^2} < 0$ on $[1, \infty)$. (Instead, you could say $f(x)$ is decreasing because as x gets larger the denominator gets larger and the numerator stays the same, so function values gets smaller.) Using partial fractions,

$$\int_1^{\infty} \frac{3}{x^2+3x}\,dx = \lim_{a\to\infty} \int_1^a \frac{1}{x} - \frac{1}{x+3}\,dx = \lim_{a\to\infty} [\ln x - \ln(x+3)]\Big|_1^a = \lim_{a\to\infty} \ln\left(\frac{x}{x+3}\right)\Big|_1^a$$

$$= \lim_{a\to\infty} \ln\left(\frac{a}{a+3}\right) - \ln\left(\frac{1}{4}\right) = \lim_{a\to\infty} \ln\left(\frac{1}{1+\frac{3}{a}}\right) + \ln 4 = \ln 1 + \ln 4 = \ln 4.$$

Since the integral converges, by the integral test the series $\sum_{n=1}^{\infty} \frac{3}{n^2+3n}$ converges.

EXAMPLE 0.0.2. Determine whether $\sum_{n=1}^{\infty} \frac{4}{n^{7/2}}$ converges.

Solution. Apply the p-series test. The series $\sum_{n=1}^{\infty} \frac{4}{n^{7/2}} = 4\sum_{n=1}^{\infty} \frac{1}{n^{7/2}}$. This is now a p-series, with $p = \frac{7}{2} > 1$. By the p-series test, the series converges.

EXAMPLE 0.0.3. Determine whether $\sum_{n=1}^{\infty} 4 \cdot \left(\frac{3}{2}\right)^n$ converges.

Solution. Apply the geometric series test. $\sum_{n=1}^{\infty} 4 \cdot \left(\frac{3}{2}\right)^n = 4(\frac{3}{2}) + 4(\frac{3}{2})^2 + 4(\frac{3}{2})^3 + \cdots$. So $|r| = |\frac{3}{2}| > 1$. By the geometric series test, the series diverges.

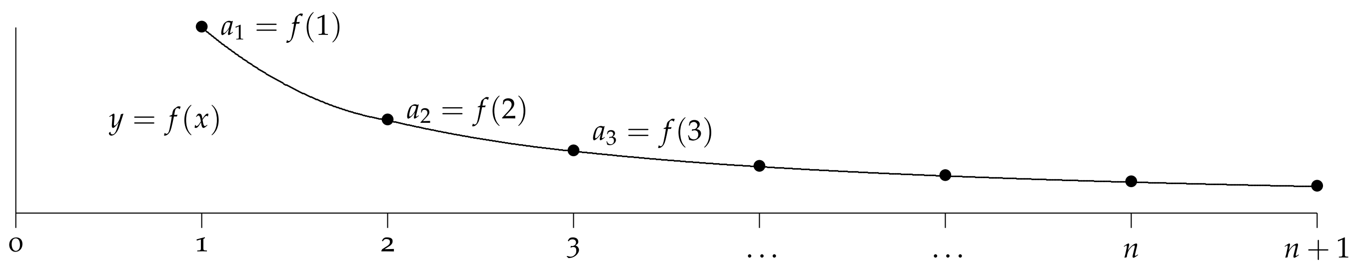
Series and Integrals: The Integral Test

The integral test is great! It combines a number of key concepts in the course: Riemann sums, improper integrals, sequences, and series. Yet it is a very intuitive result. Here's the idea. **Assume that:**

(1) We have a series $\sum_{n=1}^{\infty} a_n$ where a_n is a function $f(n)$ defined on the positive integers.

(2) Assume that the corresponding function $f(x)$ of the continuous variable x on the interval $[1,\infty)$ is **positive**, **continuous**, and **decreasing**.

For example, if we started with the series $\sum \frac{1}{n}$, then $f(x) = \frac{1}{x}$ is positive, continuous, and decreasing on $[1,\infty)$.



Each • indicates a point of the sequence $\{a_n\}$ with the graph of the corresponding function $f(x)$ for $x \geq 1$. Notice that $f(x)$ is positive, decreasing, and continuous.

We approximate the area under $f(x)$ on the interval $[1, n+1]$ by using both a **left**-hand Riemann sum, Left(n) and a **right**-hand Riemann sum, Right(n) with $\Delta x = \frac{(n+1)-1}{n} = 1$.

☞ Because f is decreasing, Left(n) is an ____________ estimate for $\int_1^{n+1} f(x)\,dx$.

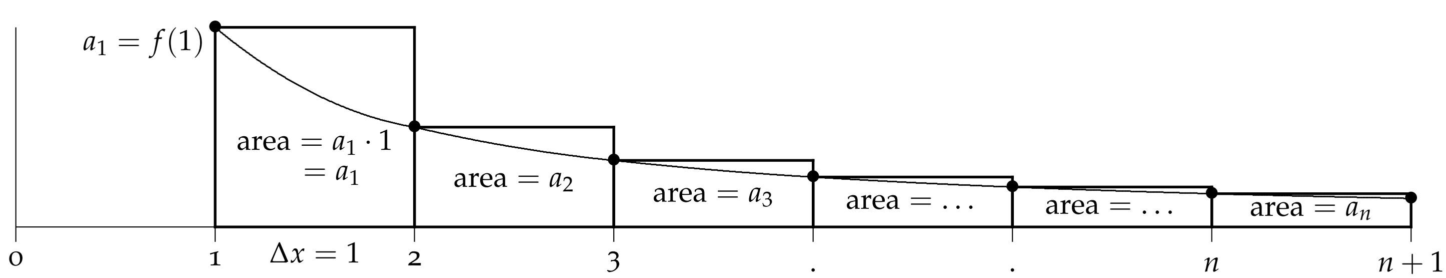
☞ Because f is decreasing, Right(n) is an ____________ estimate for $\int_1^{n+1} f(x)\,dx$.

☞ So: Right(n) $\qquad \int_1^{n+1} f(x)\,dx \qquad$ Left(n).

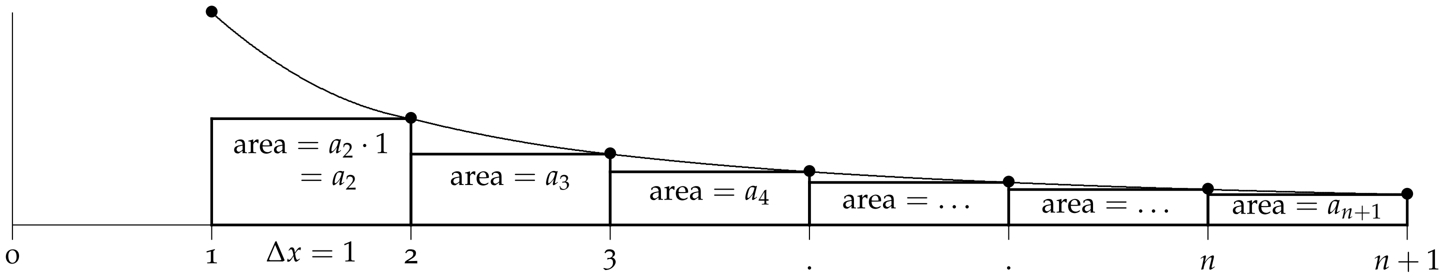
Study the graph of Left(n) below: Each rectangle has width $\Delta x = 1$ and height $f(k) = a_k$, so the area of the kth rectangle is just a_k. So the Riemann sum is

$$\text{Left}(n) = \sum_{k=1}^{n} f(k)\Delta x = \sum_{k=1}^{n} f(k) = \sum_{k=1}^{n} a_k = S_n$$

That is, the left-hand Riemann sum is just the nth partial sum of the series. **Wow!** Life does not get any better than this!!



Study the figure below and write out the sum $\text{Right}(n) = \sum\limits_{k=} f(\ \ )\Delta x = \sum\limits_{k=}$



We know that

$$\text{Right}(n) < \int_1^{n+1} f(x)\,dx < \text{Left}(n)$$

or

$$\sum_{k=2}^{n+1} a_k < \int_1^{n+1} f(x)\,dx < \sum_{k=1}^{n} a_k$$

Taking the limit as $n \to \infty$ we get the improper integral in the middle:

$$\sum_{k=2}^{\infty} a_k < \int_1^{\infty} f(x)\,dx < \sum_{k=1}^{\infty} a_k \tag{1}$$

Now suppose that the improper integral diverges (goes to ∞). Since the full series is even bigger, the series must diverge, too.

☞ Take-home Message 1: If the improper integral **diverges** to infinity, so does the corresponding series!

On the other hand, if the series $\sum\limits_{k=1}^{n} a_k$ diverges, so does series $\sum\limits_{k=2}^{n} a_k$ since the first few terms of a series don't matter for convergence or divergence. But in equation (1), the improper integral $\int_1^{\infty} f(x)\,dx$ is bigger than $\sum\limits_{k=2}^{n} a_k$, so the improper integral must diverge, too.

☞ Take-home Message 2: Thus, if the series **diverges**, so does the improper integral!!

We combine the two take-home messages into the following neat theorem.

Theorem: The Integral Test. Given $\sum\limits_{n=1}^{\infty} a_n$ and a **positive**, **continuous**, and **decreasing** function $f(x)$ such that $f(n) = a_n$.

☞ Then either both $\sum\limits_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$ diverge or both converge.

Note 1: To apply the theorem, it is sufficient for $f(x)$ to be (*eventually*) positive and decreasing on some interval of the form $[a, \infty)$ where $a > 1$. It is the infinite tail of the the improper integral or the series that determines convergence or divergence, not the first few terms.

Note 2: If the integral converges, so does the series, but their values will be different.