My Office Hours: M & W 2:30–4:00, Tu 2:00–3:30, & F 1:30–2:30 or by appointment. **Math Intern:** Sun: 2:00–5:00, 7:00–10pm; Mon thru Thu: 3:00–5:30 and 7:00–10:30pm in Lansing 310. Website: `http://math.hws.edu/~mitchell/Math131F15/index.html`.

☕ *Practice.* **Review** all of Section 8.5. The Comparison test is new today. The Root Test and Limit Comparison Test were new the Friday before Thanksgiving. Begin to read Section 8.6 on Alternating Series 649–652. Skip the subsection on remainders. But do read pages 654–655 on Absolute Convergence. See the nice summary chart on page 656.

1. (*a*) Ratio and Root Tests. **Try** page 647ff #15, 21, 23.

(*b*) Comparison and Limit Comparison Tests. **Try** page 648 #27, 29, 33, 35 and 44.

For #44 use the comparison test. What do you know about the size of $\sin^2 k$?

Nine Tests

1. Ratio Test. Assume that $\sum_{n=1}^{\infty} a_n$ is a series with **positive** terms and let $r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

1. If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
2. If $r > 1$ or $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > \infty$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $r = 1$, then the test is inconclusive. The series may converge or diverge.

2. Root Test. Assume that $\sum_{n=1}^{\infty} a_n$ is a series with **positive** terms and let $r = \lim_{n\to\infty} \sqrt[n]{a_n}$.

1. If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
2. If $r > 1$ or $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > \infty$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $r = 1$, then the test is inconclusive. The series may converge or diverge.

3. Limit Comparison Test. Assume that $a_n > 0$ and $b_n > 0$ for all n (or at least all $n \geq k$) and that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.

(1) If $0 < L < \infty$ (i.e., L is a positive, *finite* number), then either $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ *both* converge or *both* diverge.

(2) If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(3) If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

4. Direct Comparison Test. Assume $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

(*a*) If $0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(*b*) If $0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

If the bigger series converges, so does the smaller series.
If the smaller series diverges, so does the bigger series.

5. The p-series Test. The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1. \end{cases}$

6. The nth term test for Divergence. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges. (If $\lim_{n\to\infty} a_n = 0$, this test is useless.)

7. The Geometric Series Test.

(a) If $|r| < 1$, then the geometric series $\sum_{n=0}^{\infty} ar^n$ converges to $\dfrac{a}{1-r}$.

(b) If $|r| \geq 1$, then the geometric series $\sum_{n=0}^{\infty} ar^n$ diverges.

8. The Integral Test. If $f(x)$ is a **positive**, **continuous**, and **decreasing** for $x \geq 1$ and $f(n) = a_n$, then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$ either both converge or both diverge.

9. The Alternating Series Test. Assume $a_n > 0$. The alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ converges if the following two conditions hold:

(a) $\lim_{n\to\infty} a_n = 0$

(b) $a_{n+1} \leq a_n$ for all n, i.e., a_n is decreasing.

Hand In—Be Especially Neat and Careful. Use the model.

Carefully determine whether the following series converge or diverge. The grading will be very strict on these. Follow the model examples: (a) State the test you will use, (b) give a brief reason why you selected it, (c) state what you need to check before you can apply the test and do the check, (d) do any necessary calculations, and (e) interpret the results.

Model 1: Determine whether the series $\sum_{n=1}^{\infty} \dfrac{2}{n-\frac{1}{2}}$ converges.

Solution. (a) Use Limit Comparison. (b) The given series looks a lot like the p-series $\sum_{n=1}^{\infty} \frac{1}{n}$. (c) The terms $a_n = \frac{2}{n-\frac{1}{2}}$ and $b_n = \frac{1}{n}$ are always positive. (d) Notice that

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2}{n-\frac{1}{2}} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{2}{1-\frac{1}{2n}} = 2.$$

(e) The p-series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges ($p = 1 \leq 1$), and $0 < L < \infty$, so $\sum_{n=1}^{\infty} \frac{2}{n-\frac{1}{2}}$ diverges by the Limit Comparison Test.

☞Direct Comparison or the Integral Test are also possible.

Model 2: Determine whether the series $\sum_{n=1}^{\infty} \dfrac{2^n n!}{n^n}$ converges.

Solution. (a) Use the Ratio Test. (b) The given series has factorials. (c) The terms $a_n = \frac{2^n n!}{n^n}$ are positive. (d) Notice that

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n n!} = \lim_{n\to\infty} \frac{2(n+1)n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{2n^n}{(n+1)^n} = \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} 2\left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{2}{e} < 1.$$

(e) Since $r < 1$, the series converges by the Ratio Test.

Model 3: Determine whether the series $\sum_{n=1}^{\infty} \dfrac{\cos^2(n)}{n^2+6n}$ converges.

Solution. (a) Use Direct Comparison. (b) The given series looks a lot like the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. (c) The terms $a_n = \frac{\cos^2(n)}{n^2+6n}$ and $b_n = \frac{1}{n^2}$ are always positive. (d) Notice that $0 < \frac{\cos^2(n)}{n^2+6n} < \frac{1}{n^2}$ because $\cos^2 < 1$ and the denominator $n^2 + 6n > n^2$. (e) The p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges ($p = 2 > 1$), so $\sum_{n=1}^{\infty} \dfrac{\cos^2(n)}{n^2+6n}$ is SMALLER than the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, so it also converges by the Comparison Test.

☞Limit Comparison is not possible because $\cos^2(n)$ does not have a limit.

1. Using the approach above, do page 648 #21, 28, 34 (LC with $\sum_{n=1}^{\infty} \frac{1}{3^n}$), 38, and 65.

2. Bonus. Read ahead Using the approach above, do page 657 #12.