42. Determine whether the series $\sum_{k=1}^{\infty}\left(\frac{k^2}{2k^2+1}\right)^k$ converges.

Solution. (a) Use the Root Test. (b) The given series has powers. (c) The terms $a_k = (\frac{k^2}{2k^2+1})^k$ are positive. (d) Notice that

$$r = \lim_{k\to\infty}\sqrt[k]{\left(\frac{k^2}{2k^2+1}\right)^k} = \lim_{k\to\infty}\frac{k^2}{2k^2+1} = \lim_{k\to\infty}\frac{1}{2+\frac{1}{k^2}} = \frac{1}{2} < 1.$$

(e) Since $r < 1$, the series converges by the Root Test.

44. Does $\sum_{k=1}^{\infty}\frac{\sin^2 k}{k^2}$ converge?

Solution. (a) Use Direct Comparison. (b) The given series looks a lot like the p-series $\sum_{k=1}^{\infty}\frac{1}{k^2}$. (c) The terms $a_k = \frac{\sin^2 k}{k^2}$ and $b_k = \frac{1}{k^2}$ are always positive. (d) Since $0 \le \sin^2 k \le 1$, it follows that $0 < \frac{\sin^2 k}{k^2} < \frac{1}{k^2}$. (e) The p-series $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges ($p = 2 > 1$), so $\sum_{k=1}^{\infty}\frac{\sin^2 k}{k^2}$ is SMALLER than the convergent series $\sum_{k=1}^{\infty}\frac{1}{k^2}$, so it also converges by the Direct Comparison Test.

☞Limit Comparison is not possible because $\sin^2 n$ does not have a limit as $n\to\infty$.

16. Determine whether the series $\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2+10}$ converges.

Solution. (a–b) Use the Alternating Series test since the series is alternating. (c) Here $a_k = \frac{1}{k^2+10}$. Check the two conditions.

(1) Decreasing? Let $f(x) = \frac{1}{x^2+10}$. Then $f'(x) = \frac{-2x}{(x^2+10)^2} < 0$ on $[1,\infty)$ so the function and the sequence are decreasing.

(2) And $\lim_{k\to\infty} a_k = \lim_{k\to\infty}\frac{1}{k^2+10} = 0$.

(e) By the Alternating Series Test, $\sum_{k=1}^{\infty}\frac{\sin^2 k}{k^2}$ converges.

14. Determine whether the series $\sum_{k=1}^{\infty}(-1)^k\left(1+\frac{1}{k}\right)^k$ converges.

Solution. (a–b) Use the Alternating Series test since the series is alternating. (c–d) Here $a_k = \left(1+\frac{1}{k}\right)^k$. Check the two conditions. You should recognize immediately a key limit here

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = e \neq 0.$$

(e) The Alternating Series Test does not apply. BUT since $\lim_{k\to\infty} a_k \neq 0$, the Divergence Test shows that the series diverges.

18. Determine whether the series $\sum_{k=2}^{\infty}(-1)^k\frac{\ln k}{k^2}$ converges.

Solution. (a–b) Use the Alternating Series test since the series is alternating. (c) Here $a_k = \frac{\ln k}{k^2}$. (d) Check the two conditions.

(1) Decreasing? Let $f(x) = \frac{\ln x}{x^2}$. Then $f'(x) = \frac{\frac{1}{x}\cdot x^2 - 2x\ln x}{x^4} = \frac{1-2\ln x}{x^3} < 0$ on $[2,\infty)$ so the function and the sequence are decreasing.

(2) And

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty}\frac{\ln k}{k^2} = \lim_{x\to\infty}\frac{\ln x}{x^2} \overset{\text{l'Ho}}{=} \lim_{x\to\infty}\frac{\frac{1}{x}}{2x} = \lim_{x\to\infty}\frac{1}{2x^2} = 0.$$

(e) By the Alternating Series Test, $\sum_{k=2}^{\infty}(-1)^k\frac{\ln k}{k^2}$ converges.

3. Determine where the following argument is wrong. Consider $\sum_{n=1}^{\infty} \frac{2}{n^2+n}$.

Solution. (a-b) Use the Direct Comparison Test with $\sum_{n=1}^{\infty} \frac{2}{n}$. (c) Both series have positive terms. (d) Since $n^2+n > n$, so taking reciprocals changes the direction of the inequality and we have

$$0 < \tfrac{1}{n^2+n} < \tfrac{1}{n} \quad \text{so} \quad 0 < \tfrac{2}{n^2+n} < \tfrac{2}{n}.$$

(e) Since $\sum_{n=1}^{\infty} \frac{2}{n}$ diverges by the p-series test ($p=1$), then $\sum_{n=1}^{\infty} \frac{2}{n^2+n}$ diverges by the Direct Comparison Test.

Answer

The set-up of the argument is wrong. The comparison should be with $\sum_{n=1}^{\infty} \frac{2}{n^2}$. (And this can be used to show that the series actually converges.) While it is true that

$$0 < \tfrac{2}{n^2+n} < \tfrac{2}{n},$$

the Direct Comparison Test says nothing about a series smaller than a known diverging series. When comparing to a known diverging series we must take care to show that the terms of the unknown series are LARGER than the terms in the known diverging series. In that way, the unknown series must also diverge.