# Math 131 Lab 1

**1. 2-minute Review.** These should be quick. Remember you can always check that an antiderivative is correct by taking its derivative.

**a)** $\int 2+6x-9x^3\,dx$ **b)** $\int 4x^{-1}-e^{3x}\,dx$ **c)** $\int \frac{\cos x}{2}\,dx$ **d)** $\int c\,dx$ **e)** $\int \sqrt[3]{x}\,dx$

**2.** Evaluate these limits (use L'Hopital's Rule, if appropriate). First classify the indeterminate type of each, e.g., $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\cdot\infty$.

**a)** $\lim_{x\to 0}\frac{\arctan 4x}{\sin 2x}$ **b)** $\lim_{x\to 0}\frac{e^x-1-x}{x^2}$ **c)** $\lim_{x\to 0^+} x\ln x$ **d)** $\lim_{x\to\infty}\frac{x\ln x}{x^2+1}$ **e)** $\lim_{x\to 0}\frac{2^x-4^x}{x+5x^2}$

**3. Sigma Notation.** Recall from class yesterday how sigma notation is used:

Sum ends with $i=n$ ↓

Sigma tells us to sum. → $\sum_{i=m}^{n} a_i$ ← Tells us what formula to sum. Often a function of $i$.

↑ Sum starts with $i=m$

Example: $\sum_{i=1}^{5} 2i = 2(1)+2(2)+2(3)+2(4)+2(5)=30.$

Translate each of the following as in the example above:

**a)** $\sum_{i=1}^{4} i^2$ **b)** $\sum_{j=3}^{5}\cos(j\pi x)$ **c)** $\sum_{j=0}^{3} j^3+2j$

Now write each of the following sums using sigma notation.

**d)** $4+6+8+10+12$ **e)** $3+9+27+81+243$

**f)** $-1+1-1+1-1+1$ **g)** $0+1+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}$

**4.** Use summation properties and formulæ (see Theorem 5.1, p 340) to evaluate the following general sums. Your answer will be in terms of $n$. **Simplify all answers**.

**a)** $\sum_{i=1}^{n}\frac{i^2+1}{n^3}$ **b)** $\sum_{i=1}^{n}\left(10-\frac{i}{n}\right)\left(\frac{1}{n}\right)$

**5.** Evaluate the following limits using your answers to above. [Do not redo the work above; use your simplification.] Use proper limit notation.

**a)** $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^2+1}{n^3}$ **b)** $\lim_{n\to\infty}\sum_{i=1}^{n}\left(10-\frac{i}{n}\right)\left(\frac{1}{n}\right)$

**6. a)** Find the **derivative** of $y=x\ln x-x$.

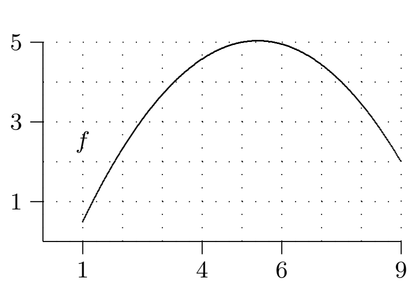
**b)** Determine $\int \ln x\,dx$. Hint: Look up.

**7.** Find the general antiderivatives of these somewhat more interesting functions. Remember you can check your answers by differentiating. For (e), divide first. Remember '+$c$'.

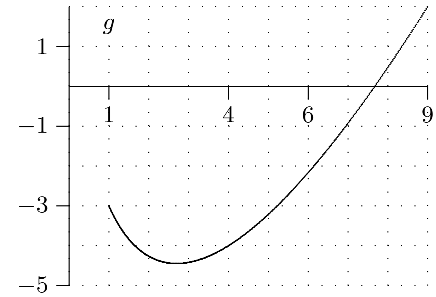
**a)** $\int 6x^{-1/2}-4\sec^2 x\,dx$ **b)** $\int \sqrt[5]{x^2}\,dx$ **c)** $\int \frac{4}{7\sqrt[3]{x^4}}\,dx$ **d)** $\int \sin 2x\,dx$

**e)** $\int \frac{9x^3-2x+x^{1/2}}{x^2}\,dx$ **f)** $\int \frac{5}{\sqrt{1-x^2}}\,dx$ **g)** $\int \sec(3x)\tan(3x)\,dx$ **h)** $\int \cos^2 x+\sin^2 x\,dx$ EZ!

**8.** **a)** From Calculus I, the Extreme Value Theorem says that *any continuous function on a closed interval* $[a, b]$ *has both maximum and a minimum value on the interval.* The two graphs below are both continuous. For each of the closed subintervals listed, estimate the max and min values of the functions $f$ and $g$.




| **Interval** | $[1, 4]$ | $[4, 6]$ | $[6, 9]$ |
|---|---|---|---|
| **Max** $f$ | | | |
| **min** $f$ | | | |
| **Max** $g$ | | | |
| **min** $g$ | | | |




**b)** For the function $f$, draw the three rectangles whose bases are the intervals along the $x$-axis and whose heights are the min values of $f$ on each interval. What is the sum of their areas? Is this sum an over or underestimate the area under the curve? Why?

**c)** For the function $g$, draw the three rectangles whose bases are the intervals along the $x$-axis and whose heights are the MAX values of $g$ on each interval. What is the sum of their areas? Over or underestimate? Why?

**9.** Determine the particular function $f(x)$ such that $f'(x) = 6x^2 + 1$ and $f(1) = 4$. First find the general antiderivative, then evaluate $c$ by using the given value. There should be no unknown constant term "c" in your final answer.

**10.** Suppose that $f''(x) = 2\cos x$. If $f'(0) = 1$ and $f(0) = 2$, what is the function $f(x)$? Hint: Do antidifferentiation twice.

**11.** Sometimes when we have no formula for a function we are forced to do graphical antidifferentiation. Let $F(x)$ be the antiderivative of $f(x)$ on $[-3, 4]$, where $f$ is the function graphed on the *left* below. Since $F$ is an antiderivative of $f$, then $F' = f$. Use this relationship to answer the following questions.

**a)** Where is $F'$ positive? Negative? Use $F'$ to determine the interval(s) where $F$ increasing. Decreasing.

**b)** At what point(s), if any, does $F$ have a local max? Min?

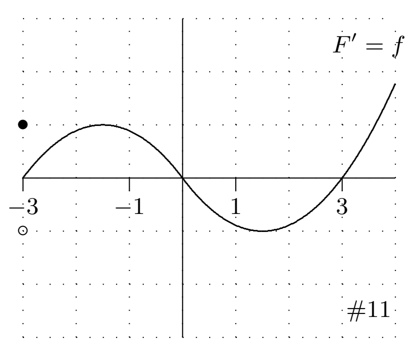
**c)** Determine where $F''$ is positive and negative. On what interval(s) is $F$ concave up? Down?

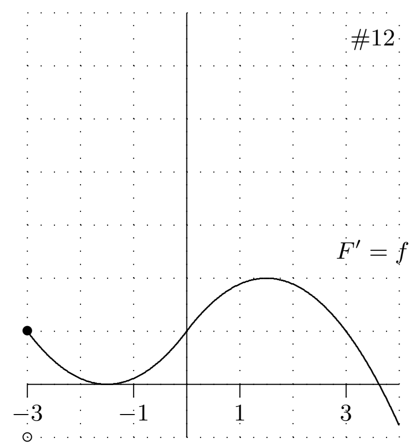
**d)** Does $F$ have any points of inflection?

**e)** Assume $F$ passes through the point $(-3, 1)$ indicated with a $\bullet$; draw a potential graph of $F$.

**f)** Assume, instead, that $F$ passes through $(-3, -1)$ indicated by a $\circ$; draw a graph of $F$.

**g)** What is the relationship between the two graphs you've drawn?




**12.** If time allows: Repeat Problem 11 for the graph on the right above.

**13.** A sea otter ingests a pollutant and immediately begins to excrete it at a rate of

$$\frac{dA}{dt} = \frac{k}{1 + t^2},$$

where $A(t)$ is the amount of the pollutant (in mg) remaining after $t$ days and $k$ is an unknown constant. If the initial amount of the pollutant ingested is $A(0) = 20$ mg and a day later there is $A(1) = 15$ mg left, find the function $A(t)$. How much of the pollutant remains after 5 days?

**Answers to Math 131 Lab 1** **Extra credit if you find a typo. E-mail me.**

**1.** The antiderivatives are:

**a)** $2x+3x^2-\frac{9}{4}x^4+c$ **b)** $4\ln|x|-\frac{1}{3}e^{3x}+c$

**c)** $\frac{1}{2}\sin x+c$ **d)** $cx+d$ **e)** $\frac{3}{4}x^{4/3}+c$

**2. a)** Form $\frac{0}{0}$: $\lim\limits_{x\to 0}\dfrac{\arctan 4x}{\sin 2x}\overset{\text{l'Ho}}{=}\lim\limits_{x\to 0}\dfrac{\frac{4}{1+16x^2}}{2\cos 2x}=\dfrac{\frac{4}{1}}{2}=2.$

**b)** Form $\frac{0}{0}$: $\lim\limits_{x\to 0}\dfrac{e^x-1-x}{x^2}\overset{\text{l'Ho}}{=}\lim\limits_{x\to 0}\dfrac{e^x-1}{2x}\overset{\text{l'Ho}}{=}\lim\limits_{x\to 0}\dfrac{e^x}{2}=\dfrac{1}{2}.$

**c)** Form $0\cdot\infty$: $\lim\limits_{x\to 0^+}x\ln x=\lim\limits_{x\to 0^+}\dfrac{\ln x}{\frac{1}{x}}\overset{\text{l'Ho}}{=}\lim\limits_{x\to 0^+}\dfrac{\frac{1}{x}}{-\frac{1}{x^2}}=\lim\limits_{x\to 0^+}-\dfrac{x^2}{x}=\lim\limits_{x\to 0^+}-x=0.$

**d)** Form $\frac{\infty}{\infty}$: $\lim\limits_{x\to\infty}\dfrac{x\ln x}{x^2+1}\overset{\text{l'Ho}}{=}\lim\limits_{x\to\infty}\dfrac{1+\ln x}{2x}\overset{\text{l'Ho}}{=}\lim\limits_{x\to\infty}\dfrac{\frac{1}{x}}{2}=\dfrac{0}{2}=0.$

**e)** $\frac{0}{0}$. $\lim\limits_{x\to 0}\dfrac{2^x-4^x}{x+5x^2}\overset{\text{l'Ho}}{=}\lim\limits_{x\to 0}\dfrac{2^x\ln 2-4^x\ln 4}{1+10x}=\ln 2-\ln 4=\ln\dfrac{2}{4}=-\ln 2.$

**3.** **a)** $\displaystyle\sum_{i=1}^{4}i^2=1+4+9+16=30$ **b)** $\displaystyle\sum_{j=3}^{5}\cos(j\pi x)=\cos(3\pi x)+\cos(4\pi x)+\cos(5\pi x)$

**c)** $\displaystyle\sum_{j=0}^{3}j^3+2j=0+3+12+33=38$ **d)** $\displaystyle\sum_{j=2}^{6}2j$ **e)** $\displaystyle\sum_{j=1}^{5}3^j$ **f)** $\displaystyle\sum_{j=1}^{6}(-1)^j$ **g)** $\displaystyle\sum_{j=0}^{20}\sqrt{j}$

**4. a)** $\displaystyle\sum_{i=1}^{n}\frac{i^2+1}{n^3}=\frac{1}{n^3}\sum_{i=1}^{n}i^2+\frac{1}{n^3}\sum_{i=1}^{n}1=\frac{n(n+1)(2n+1)}{6n^3}+\frac{n}{n^3}=\frac{2n^2+3n+1}{6n^2}+\frac{1}{n^2}=\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}+\frac{1}{n^2}$

**b)** $\displaystyle\sum_{i=1}^{n}\left(10-\frac{i}{n}\right)\left(\frac{1}{n}\right)=\frac{10}{n}\sum_{i=1}^{n}1-\frac{1}{n^2}\sum_{i=1}^{n}i=\frac{10n}{n}-\frac{n(n+1)}{2n^2}=10-\frac{n+1}{2n}=10-\frac{1}{2}-\frac{1}{2n}=9.5-\frac{1}{2n}.$

**5. a)** $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^2+1}{n^3}=\lim_{n\to\infty}\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}+\frac{1}{n^2}=\frac{1}{3}+0+0+0=\frac{1}{3}$

**b)** $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(10-\frac{i}{n}\right)\left(\frac{1}{n}\right)=\lim_{n\to\infty}9.5-\frac{1}{2n}=9.5-0=9.5$

**6. a)** Product rule: $y'=1\cdot\ln x+x\cdot\frac{1}{x}-1=\ln x$. **b)** Looking at part (a): $\int\ln x\,dx=x\ln x-x+c$.

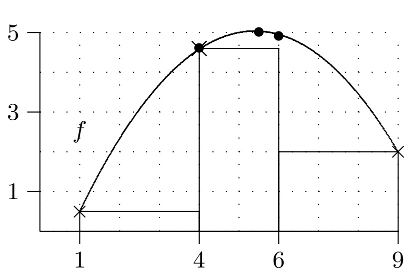
**7.** The antiderivatives are:

**a)** $12x^{1/2}-4\tan x+c$ **b)** $=\int x^{2/5}\,dx=\frac{5}{7}x^{7/5}+c$ **c)** $=\int\frac{4}{7}x^{-4/3}\,dx=-\frac{12}{7}x^{-1/3}+c$

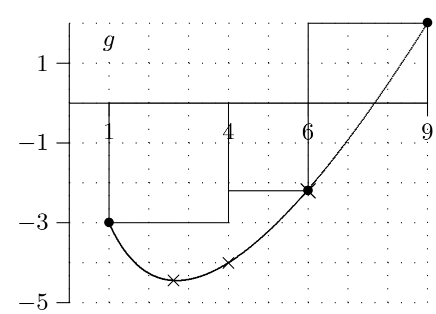
**d)** $-\dfrac{\cos(2x)}{2}+c$ **e)** $\displaystyle\int 9x-2x^{-1}+x^{-3/2}\,dx=\frac{9}{2}x^2-2\ln|x|-2x^{-1/2}+c$ **f)** $5\arcsin x+c$

**g)** $\dfrac{\sec(3x)}{3}+c$ **h)** $x+c$. Remember : $\cos^2x+\sin^2x=1$

**8. a)** Maxs are denoted with • and mins with ×. Why are some points marked as both a max and a min?




| **Interval** | $[1,4]$ | $[4,6]$ | $[6,9]$ |
|---|---|---|---|
| **Max** $f$ | 4.6 | 5 | 4.8 |
| **min** $f$ | 0.5 | 4.5 | 2 |
| **Max** $g$ | $-3$ | $-2.2$ | 2 |
| **min** $g$ | $-4.4$ | $-4$ | $-2.2$ |




**b)** Area sum: $3\times 0.5+2\times 4.5+3\times 2=16.5$. Underestimate. The rectangles are below the curve.

**c)** Area sum: $3\times(-3)+2\times(-2.2)+3\times 2=16.5-7.4$. Overestimate. The rectangles are always above the curve.

**9.** If $f'(x) = 6x^2 + 1$, then $f(x) = 2x^3 + x + c$. But $f(1) = 4 = 2 + 1 + c \Rightarrow c = 1$. So $f(x) = 2x^3 + x + 1$.

**10.** If $f''(x) = 2\cos x$, then $f'(x) = \int 2\cos x\, dx = 2\sin x + c$. But $f'(0) = 2(0) + c = 1 \Rightarrow c = 1$. So $f'(x) = 2\sin x + 1$ so $f(x) = -2\cos x + x + c$. But $f(0) = -2 + 0 + c = 2$, so $c = 4$. Now $f(x) = -2\cos x + x + 4$.

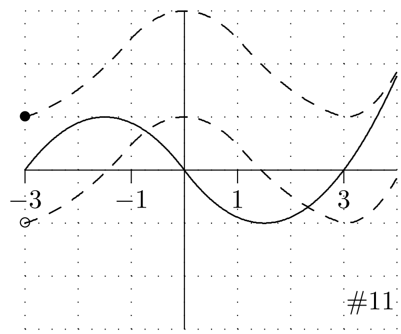
**11.** **a)** Use that $F' = f$ and $F'' = f'$. $F$ increasing means $F' = f > 0$: on $(-3, 0)$ and $(3, 4)$. $F$ decreasing means $F' = f < 0$: on $(0, 3)$.

**b)** $F$ has a local max means $F' = f$ changes from $+$ to $-$: at $x = 0$. $F$ has a local min means $F' = f$ changes from $-$ to $+$: at $x = 3$.

**c)** $F$ concave up means $F'' = f' > 0$, i.e., $f$ is increasing: $(-3, -1.5)$ and $(1.5, 4)$. $F$ concave down means $F'' = f' < 0$, i.e., $f$ is decreasing: $(-1.5, 1.5)$.

**d)** $F$ has a points of inflection when $F'' = f'$ changes sign, i.e., when $f$ changes from increasing to decreasing or *vice versa*: at $x = \pm 1.5$.

**g)** The two graphs are parallel (differ only by a constant).




**12.** This is a HW problem.

**13.** Given $\frac{dA}{dt} = \frac{k}{1+t^2}$, $A(0) = 20$, and $A(1) = 15$. So

$$A(t) = \int A'(t)\, dt = \int \frac{k}{1+t^2}\, dt = k\arctan(t) + c.$$

$A(0) = 20 = 0 + c$, so $c = 20$. Therefore, $A(t) = k\arctan(t) + 20$. Then

$$A(1) = 15 = k\arctan(1) + 20 = \frac{k\pi}{4} + 20 \to -5 = \frac{k\pi}{4} \to k = -\frac{20}{\pi}.$$

Therefore, $A(t) = -\frac{20}{\pi}\arctan t + 20$, so $A(5) = -\frac{20}{\pi}\arctan(5) + 20 \approx 11.26$ mg.