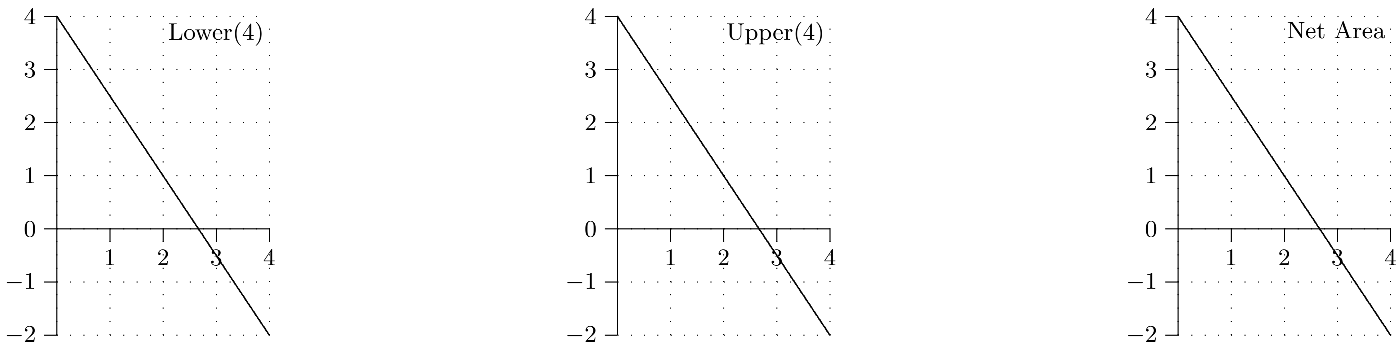
Math 131 Lab 2

Quick answers on back. Check with the TA or me!!! Remember full answers are posted online (and Class Notes).

0. 1 minute review: Find the following **derivatives** (answers on back):

a) $\dfrac{d}{dx}[\sin^6(x)]$ **b)** $\dfrac{d}{dx}[\arcsin(3x^2)$

1. a) There's no reason why in a Riemann sum $\sum_{i=1}^{n} f(x_i)\Delta x$ the function $f(x)$ needs to non-negative. Using the two graphs of f below, draw Lower(4) (the lower Riemann sum) and Upper(4) (upper sum) and evaluate each. Make sure you check that your figure is correct with me or a TA! In the final graph, shade the net area.



b) Estimate Lower(4) and Upper(4).

c) What do these sums represent geometrically?

d) The function $f(x)$ is a straight line in this problem. Figure out the equation of $f(x)$.

e) Why does the sum Lower(n) use right endpoints?

f) Set up and simplify Lower(n). Use the table below to help. And then simplify.

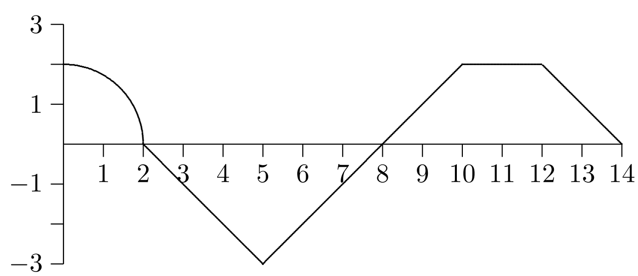
$f(x)$	$[a,b]$	$\Delta x = \frac{b-a}{n}$	$x_i = a + i\Delta x$	$f(x_i)$	$\text{Lower}(n) = \sum_{i=1}^{n} f(x_i)\Delta x$
	$[0,4]$				

g) Evaluate $\int_0^4 f(x)\,dx$ by evaluating $\lim_{n\to\infty} \text{Lower}(n)$. **Compare** your answer to the net area in the figure above.

2. a) Find the formula for regular right-hand Riemann sum Right(n) for $f(x) = x^2 - x$ on $[1,4]$. Make sure to simplify Right(n) as much as possible.

b) Evaluate the definite integral $\int_1^4 x^2 - x\,dx$ by evaluating $\lim_{n\to\infty} \text{Right}(n)$.

3. Use this graph of $y = f(x)$



to evaluate these integrals.

a) $\displaystyle\int_0^5 f(x)\,dx$ **b)** $\displaystyle\int_2^{10} f(x)\,dx$ **c)** $\displaystyle\int_5^{14} f(x)\,dx$ **d)** $\displaystyle\int_0^5 |f(x)|\,dx$

e) Is the answer to (a) or (d) the **Total Area** between f and the x-axis on $[0,5]$?

4. Suppose that $\int_{-4}^{6} f(x)\,dx = 5$, $\int_{-4}^{6} g(x)\,dx = -2$, and $\int_{-4}^{6} h(x)\,dx = 7$. Evaluate each of the following expressions by using properties of the integral. The answer to part (b) is NOT 7.

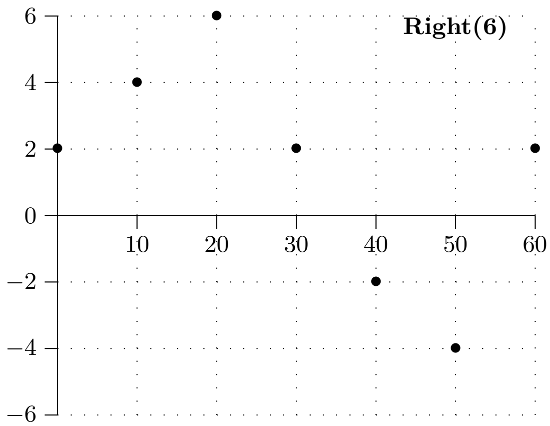
a) $\displaystyle\int_{-4}^{6} (h(x) - 4g(x))\,dx$ **b)** $\displaystyle\int_{-4}^{6} f(x) + 2\,dx$ **c)** $\displaystyle\int_{6}^{-4} 2f(x)\,dx$ **d)** $\displaystyle\int_{-4}^{-4} e^{f(x)}\,dx$ **e)** $\displaystyle\int_{0}^{10} f(x-4)\,dx$

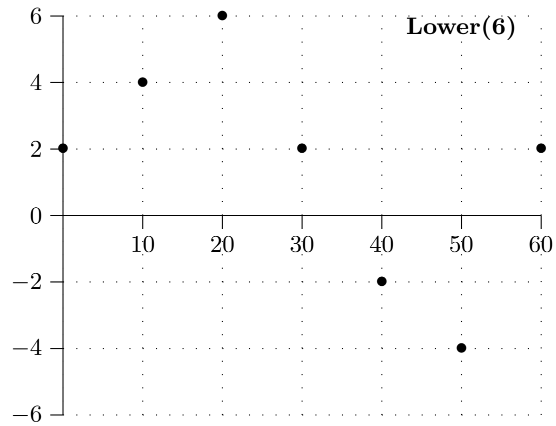
5. Interpolating Riemann sums. So far we've used functions or their graphs to generate the data for a Riemann sum. But sometimes all we have are data in table. For example we might have data that were recorded from a speedometer or a water meter. The data below are velocities that were recorded by a bicyclist every 10 seconds for a total of one minute. Use her data (on back of page) to estimate the area under her velocity curve.

a) Find Right(6). For each 10-second interval use the right endpoint for the height. What do negative heights mean?

b) Then try Lower(6). For each 10-second interval use the appropriate endpoint that gives the min value during the interval. Which is easier?

Time (s)	0	10	20	30	40	50	60
Velocity (m/s)	2	4	6	2	−2	−4	2





c) What is the physical interpretation of $v \times \Delta t$ in your sum? So a Riemann sum can represent something different than area. In this case, the area under the velocity curve represents what physical quantity?

6. Mental Adjustments. Sometimes a problem requires doing an antiderivative that is slightly different from those in our known list of antiderivatives. For example, $\int \sec(3t)\tan(3t)\,dt$ looks like it ought to have $F(t) = \sec(3t) + c$ as an antiderivative. However, if we check our answer, we see that $F'(t) = \sec(3t)\tan(3t)\cdot 3$. We are off by a factor of 3 because of the chain rule. So we need to 'adjust' the antiderivative by multiplying by $\frac{1}{3}$. Let $F(t) = \frac{1}{3}\sec(3t)+c$. Now $F'(t) = \frac{1}{3}\sec(3t)\tan(3t)\cdot 3 = \sec(3t)\tan(3t)$ which is what we want. Here's another: $\int \sec^2(-6x)\,dx = -\frac{1}{6}\tan(-6x)+c$. What is the rule for the adjustment? Do these by using a "mental adjustment" of each by an appropriate constant. (Check your antiderivatives by differentiating!)

a) $\displaystyle\int \sin(2x)\,dx$ **b)** $\displaystyle\int 3e^{-6t}\,dt$ **c)** $\displaystyle\int \sec^2(\pi x)\,dx$ **d)** $\displaystyle\int \cos\tfrac{3\theta}{2}\,d\theta$ **e)** $\displaystyle\int \frac{1}{2x}\,dx$

7. Evaluate the definite integral $\int_0^2 4x^3 - 5\,dx$ by evaluating $\lim\limits_{n\to\infty} \text{Right}(n)$.

8. a) Assume that a is a positive number. Draw the region represented by $\displaystyle\int_0^a x\,dx$.

b) Use geometry (not Riemann sums) to determine a formula for $\displaystyle\int_0^a x\,dx$ in terms of a.

c) Do the same for $\displaystyle\int_0^a x + 2\,dx$.

Brief Partial Answers

0. Use the chain rule. **(a)** $6\sin^5(x)\cos(x)$; **(b)** $\frac{6x}{\sqrt{1-9x^4}}$. Note the "square" of x^2.

1 a) Lower(4) = 1.0; (b) Upper(4) = 7.0. (f) $4 - \frac{12}{n}$. **2)** $\int_1^4 x^2 - x\,dx = \lim\limits_{n\to\infty} \text{Right}(n) = \lim\limits_{n\to\infty} \frac{27}{2} + \frac{18}{n} + \frac{9}{2n^2} = 13.5$.

3) $\pi - 4.5$; -7; 3.5; $\pi + 4.5$; d. **4)** 15; 25; −10; 0; 5. **5 a)** Right(6) = 80 m. (b) Lower(6) = −20 m.

6) $-\frac{1}{2}\cos(2x) + c$; $-\frac{1}{2}e^{-6t} + c$; $\frac{1}{\pi}\tan(\pi x) + c$; $\frac{2}{3}\sin\frac{3\theta}{2} + c$; $\frac{1}{2}\ln|2x| + c$. **7)** $\lim_{n\to\infty} 6 + \frac{32}{n} + \frac{16}{n^2} = 6$.

8) $\frac{a^2}{2}$; $2a + \frac{a^2}{2}$

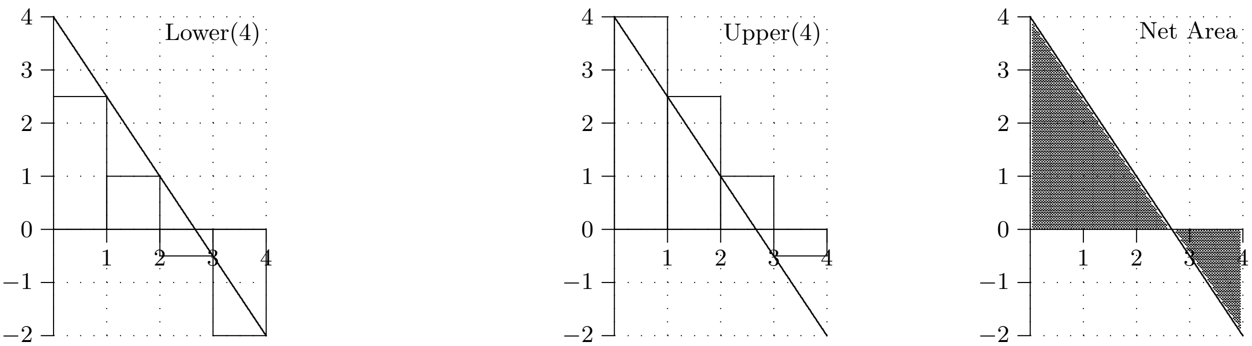
Math 131 Lab 2 Answers

See problem 3. Notice that the rectangle bases are always on the x axis.

0. Answer: The derivatives all require the chain rule. Note the "square" in (b).

a) $6\sin^5(x)\cos(x)$ **b)** $\dfrac{6x}{\sqrt{1-9x^4}}$

1. a) Note: The bases of the rectangles are always on the x axis, which may not be the bottom of the grid!!!



b) Multiplying the heights by $\Delta x = 1$ and adding gives

$$\text{Lower}(4) = 2.5 \cdot 1 + 1 \cdot 1 + (-0.5) \cdot 1 + (-2) \cdot 1 = 1.0$$

$$\text{Upper}(4) = 4 \cdot 1 + 2.5 \cdot 1 + 1 \cdot 1 + (-0.5) \cdot 1 = 7.0.$$

c) The rectangles below the axis produce 'negative' area, so the result is **net area**, that is area above the x-axis minus the area below it.

d) It passes through the points $(0, 4)$ and $(4, -2)$ so the slope is $m = \frac{4-(-2)}{0-4} = -\frac{3}{2}$. The intercept is 4 so the equation is $f(x) = -\frac{3}{2}x + 4$.

e) $f(x)$ is decreasing so the lowest point in each interval is at the right end.

f)

$f(x)$	$[a,b]$	$\Delta x = \frac{b-a}{n}$	$x_i = a + i\Delta x$	$f(x_i)$	$\text{Right}(n) = \sum_{i=1}^{n} f(x_i)\Delta x$
$-\frac{3}{2}x + 4$	$[0,4]$	$\frac{4-0}{n} = \frac{4}{n}$	$0 + \frac{4i}{n} = \frac{4i}{n}$	$-\frac{3}{2} \cdot \frac{4i}{n} + 4 = -\frac{6i}{n} + 4$	$\sum_{i=1}^{n} \left(-\frac{6i}{n} + 4\right)\frac{4}{n}$

$$\text{Lower}(n) = \sum_{i=1}^{n} \left(-\frac{6i}{n} + 4\right)\frac{4}{n} = \left(-\frac{6}{n}\right)\frac{4}{n}\sum_{i=1}^{n} i + \frac{4}{n}\sum_{i=1}^{n} 4 = -\frac{24}{n^2}\left[\frac{n(n+1)}{2}\right] + \frac{4}{n}(4n)$$

$$= -\frac{12(n+1)}{n} + 16 = -12 - \tfrac{12}{n} + 16 = 4 - \tfrac{12}{n}.$$

g) $\lim\limits_{n\to\infty} \text{Lower}(n) = \lim\limits_{n\to\infty} 4 - \frac{12}{n} = 4$. The net area is the the area of the big triangle above the x-axis minus the area of the triangle below the axis. The line crosses the x-axis at $\frac{8}{3}$, so

$$\text{net area} = \frac{1}{2} \cdot \frac{8}{3} \cdot 4 - \frac{1}{2} \cdot \frac{4}{3} \cdot 2 = \frac{16}{3} - \frac{4}{3} = 4.$$

So the net area equals $\int_0^4 -\frac{3}{2}x + 4\,dx = \lim_{n\to\infty} \text{Lower}(n) = 4$.

2. a)

$f(x)$	$[a,b]$	$\Delta x = \frac{b-a}{n}$	$x_i = a + i\Delta x$	$f(x_i)$
$x^2 - x$	$[1,4]$	$\frac{4-1}{n} = \frac{3}{n}$	$1 + \frac{3i}{n}$	$(1+\frac{3i}{n})^2 - (1+\frac{3i}{n}) = (1 + \frac{6i}{n} + \frac{9i^2}{n^2}) - (1 + \frac{3i}{n}) = \frac{9i^2}{n^2} + \frac{3i}{n}$

$$\text{Right}(n) = \sum_{i=1}^{n}\left[\frac{9i^2}{n^2} + \frac{3i}{n}\right]\cdot\frac{3}{n} = \frac{27}{n^3}\sum_{i=1}^{n} i^2 + \frac{9}{n^2}\sum_{i=1}^{n} i = \frac{27}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] + \frac{9}{n^2}\left[\frac{n(n+1)}{2}\right]$$

$$= 9\left[\frac{2n^2+3n+1}{2n^2}\right] + \frac{9n+9}{2n} = 9 + \frac{27}{2n} + \frac{9}{2n^2} + \frac{9}{2} + \frac{9}{2n} = \frac{27}{2} + \frac{18}{n} + \frac{9}{2n^2}$$

b) Since f is continuous we can just take the limit of Right(n) to get the definite integral.

$$\int_1^4 x^2 - x\,dx = \lim_{n\to\infty} \text{Right}(n) = \lim_{n\to\infty} \frac{27}{2} + \frac{18}{n} + \frac{9}{2n^2} = 13.5$$

3. The integral is 'net area.' So using quarter-circles and triangles,

a) $\int_0^5 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^5 f(x)\,dx = \frac{\pi(2)^2}{4} + \frac{1}{2}(3)(-3) = \pi - 4.5$

b) $\int_2^{10} f(x)\,dx = \int_2^8 f(x)\,dx + \int_8^{10} f(x)\,dx = \frac{1}{2}(6)(-3) + \frac{1}{2}(2)(2) = -7$

c) $\int_5^{14} f(x)\,dx = \int_5^8 f(x)\,dx + \int_8^{10} f(x)\,dx + \int_{10}^{12} f(x)\,dx + \int_{12}^{14} f(x)\,dx = \frac{1}{2}(3)(-3) + \frac{1}{2}(2)(2) + (2)(2) + \frac{1}{2}(2)(2) = 3.5$

d) $\int_0^5 |f(x)|\,dx = \int_0^2 f(x)\,dx + \int_2^5 -f(x)\,dx = \frac{\pi(2)^2}{4} + \frac{1}{2}(3)(3) = \pi + 4.5$

e) (d), all regions are treated as positive area.

4. Use integral properties. In (b) $\int_{-4}^{6} 2\,dx$ is a rectangle with height 2 and base from $x = -4$ to 6.

a) $\displaystyle\int_{-4}^{6}(h(x) - 4g(x))\,dx = \int_{-4}^{6}(h(x)\,dx - 4\int_{-4}^{6} g(x))\,dx = 7 - 4*(-2) = 15$

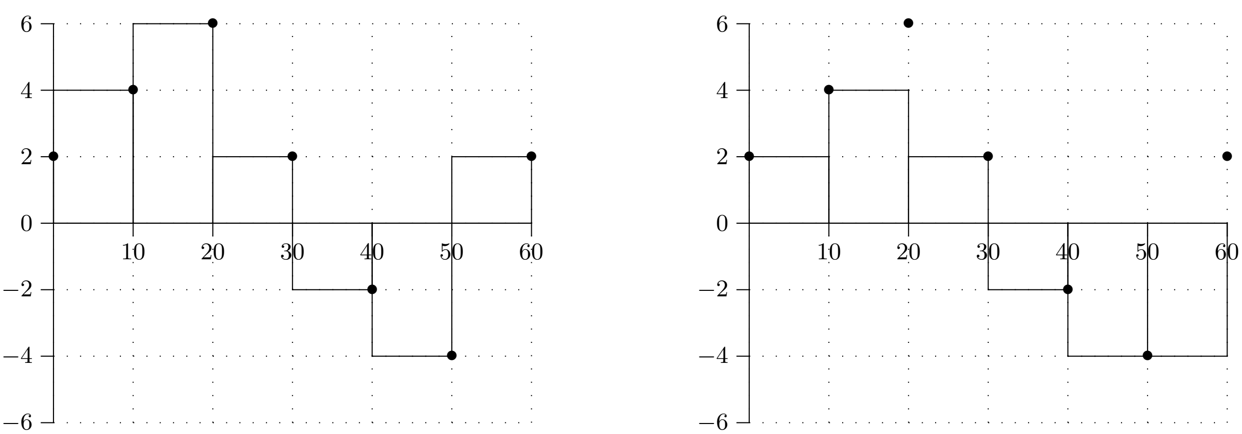
b) $\displaystyle\int_{-4}^{6} f(x) + 2\,dx = \int_{-4}^{6} f(x)\,dx + \int_{-4}^{6} 2\,dx = 5 + 2(b-a) = 5 + 2(6-(-4)) = 25$

c) $\displaystyle\int_{6}^{-4} 2f(x)\,dx = -\int_{-4}^{6} 2f(x)\,dx = -2\int_{-4}^{6} f(x)\,dx = -2\cdot 5 = -10$

d) $\displaystyle\int_{-4}^{-4} e^{f(x)}\,dx = 0$ same endpoints

e) $\displaystyle\int_{0}^{10} f(x-4), dx = \int_{-4}^{6} f(x), dx = 5$ graph was shifted

5.



a) $\Delta x = 10$. Use the right-hand endpoints

$$\text{Right}(6) = 4\cdot 10 + 6\cdot 10 + 2\cdot 10 - 2\cdot 10 - 4\cdot 10 + 2\cdot 10 = 80 \text{ m.}$$

b) For Lower(n) pick the smallest velocity in each 10-second interval.

$$\text{Lower}(6) = 2\cdot 10 + 4\cdot 10 + 2\cdot 10 - 2\cdot 10 - 4\cdot 10 - 4\cdot 10 = -20 \text{ m.}$$

c) Velocity × time equals distance travelled. Area under the curve represents the NET distance travelled by the cyclist.

6. Multiply the antiderivative by the reciprocal of the constant on the "inside" of the function (to reverse the chain rule):

a) $\int \sin(2x)\,dx = -\frac{1}{2}\cos(2x) + c$

b) $\int 3e^{-6t}\,dt = -\frac{3}{6}e^{-6t} + c = -\frac{1}{2}e^{-6t} + c$

c) $\int \sec^2(\pi x)\,dx = \dfrac{1}{\pi}\tan(\pi x) + c$

d) $\int \cos\frac{3\theta}{2}\,d\theta = \frac{2}{3}\sin\frac{3\theta}{2} + c$

e) $\int \dfrac{1}{2x}\,dx = \frac{1}{2}\ln|2x| + c$

7. Do the preliminary calculations:

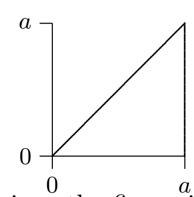
$f(x)$	$[a, b]$	$\Delta x = \frac{b-a}{n}$	$x_i = a + i\Delta x$	$f(x_i)$
$4x^3 - 5$	$[0, 2]$	$\frac{2-0}{n} = \frac{2}{n}$	$\frac{2i}{n}$	$4\left(\frac{2i}{n}\right)^3 - 5$

$$\text{Right}(n) = \sum_{i=1}^{n}\left[4\left(\frac{2i}{n}\right)^3 - 5\right]\cdot\frac{2}{n} = \frac{64}{n^4}\sum_{i=1}^{n} i^3 - \frac{10}{n}\sum_{i=1}^{n} 1 = \frac{64}{n^4}\left[\frac{n^2(n+1)^2}{4}\right] - \frac{10}{n}\cdot n$$

$$= 16\left[\frac{n^2+2n+1}{n^2}\right] - 10 = 16 + \frac{32}{n} + \frac{16}{n^2} - 10 = 6 + \frac{32}{n} + \frac{16}{n^2}$$

Since f is continuous we can just take the limit of Right(n) to get the definite integral.

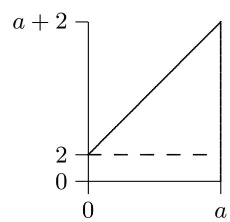
$$\int_0^2 4x^3 - 5\,dx = \lim_{n\to\infty} \text{Right}(n) = \lim_{n\to\infty} 6 + \frac{32}{n} + \frac{16}{n^2} = 6.$$



8. a) The region is a right triangle with base and height both equal to a, as shown.

b) Since $\int_0^a x\,dx$ represents the net area (here the entire area is positive), then $\int_0^a x\,dx = \dfrac{a^2}{2}$ since the figure is a right triangle.

c) The region is a right trapezoid, as shown.



The area is the sum of a rectangle and triangle, so

$\int_0^a x\,dx = \text{Net Area} = 2a + \dfrac{a^2}{2}$ since the figure is a right triangle.