Math 131 Lab 7. **Make sure to get through #1–8**

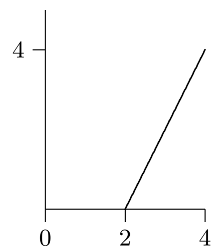
1. Find the arc length of $y = 2\ln x - \frac{1}{16}x^2 - 2$ on the interval $[1,4]$. Simplify the integrand! (Answer: $2\ln 4 + \frac{15}{16}$)

2. Find the arc length of $y = \frac{1}{8}x^4 + \frac{1}{4}x^{-2}$ on the interval $[1,2]$. Simplify the integrand! (Answer: 33/16)

3. (Set up now, do later.) Find the arc length of the *catenary* curve $y = 2e^{x/4} + 2e^{-x/4}$ on the interval from -1 to 1. (Answer: $4[e^{1/4} - e^{-1/4}]$)

4. Work ahead. Let R be the region in the first quadrant bounded by the x-axis, the y-axis, and the curve $y = 4 - x^2$. Rotate the R around the y-axis to form a silo (tank). If the silo is filled with wheat ($D = 100$ lbs per cu. ft) how much work is done in raising the wheat to the top of the silo. (Ans: $\frac{6400}{3}\pi$ ft lbs.)

5. A tank in the form of a truncated cone is formed by rotating the segment between $(2,0)$ and $(4,4)$ around the y-axis. It is filled with sludge (density 80 lbs/ft^3). If the sludge is pumped 3 feet above the tank into a tank truck, how much work was required?



6. [From an exam] Ants excavate a chamber underground that is described as follows: Let S be the region in the fourth quadrant enclosed by $y = -\sqrt{x}$, $y = -1$, and the y-axis; revolve S around the y-axis.

a) Find the volume of the chamber using the shell method. (Ans: $\pi/5$)

b) Suppose that the chamber contained soil which weighed 50 lbs per cubic foot. How much work did the ants do in raising the soil to ground level? WeBWorK problem.

7. A small farm elevated water tank is in the shape obtained from rotating the region in the first quadrant enclosed by the curves $y = 10 - \frac{1}{2}x^2$, $y = 8$, and the y-axis about the y-axis.

a) Find the work "lost" if the water (62.5 lbs/ft^3) leaks onto the ground from a hole in the bottom of the tank. (Answer: $-6500\pi/3$ ft-lbs.)

b) Find the work "lost" if the water leaks onto the ground from a hole in the side of the tank at height 9 feet. (Answer: $-1750\pi/3$ ft-lbs.)

8. Here are three integrals that require different solution techniques including by parts. (Answers on back.)

a) $\int (x+2)e^{2x}\,dx$ **b)** $\int xe^{x^2}\,dx$ **c)** $\int x\sec^2 x\,dx$ **d)** $\int x\ln x\,dx$

9. Recall that the exponential and natural log functions are inverses and so they "undo" each other. Consequently, $e^{\ln a} = a$. We can use this idea to determine another type of integral. Suppose that $b > 0$ and we want to determine $\int b^x\,dx$. We know $\int e^x\,dx = e^x + c$. We can can rewrite b^x as

$$b^x = e^{lnb^x} = e^{x\ln b}.$$

Remember that $\ln b$ is just a constant, so using a 'mental adjustment', we find that

$$\int b^x\,dx = \int e^{x\ln b}\,dx = \frac{1}{\ln b}e^{x\ln b} + c.$$

This simplifies to

$$\boxed{\int b^x\,dx = \frac{b^x}{\ln b} + c.}$$

The chain rule version of this is

$$\boxed{\int b^u\,du = \frac{b^u}{\ln b} + c,}$$

where u is a function of x. Use these new formulas to determine the following. (Answers below.)

a) $\int 6^x\,dx$ **b)** $\int 4\cdot 5^x\,dx$ **c)** $\int 3^{\cos(2x)+1}\sin(2x)\,dx$ **d)** $\int (x^2+1)2^{x^3+3x}\,dx$

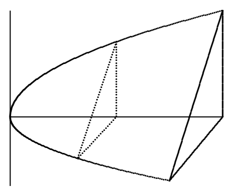
10. You are the professor. Recall this result from Homework: If a and n are a non-zero real numbers, then

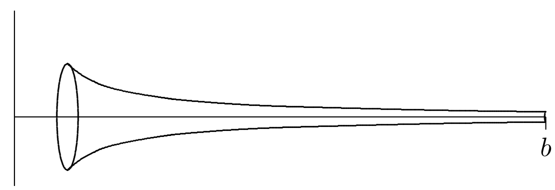
$$1+\left(ax^n - \frac{1}{4a}x^{-n}\right)^2 = \left(ax^n + \frac{1}{4a}x^{-n}\right)^2.$$

a) Design an arc length problem based on this. For your choice of a and n, you will need to figure out a function $f(x)$ so that $f'(x) = ax^n - \frac{1}{4a}x^{-n}$.

b) Determine the arc length of your function $f(x)$ on $[1,2]$. (Ans: For your choice of a and n: $\frac{2^{n+1}a}{n+1} - \frac{1}{4a(n-1)2^{n-1}} - \frac{a}{n+1} + \frac{1}{4a(n-1)}$.)

11. Non-rotation problem. A crystal prism is 9 cm long. Its cross-sections are isosceles right triangles with heights formed by the curve $y = 2\sqrt{x}$. Find the volume of the prism. (See figure on left below.) (Answer: 81cc)





12. a) Here's a fun problem to think about. First, let R be the region under the curve $y = f(x) = \frac{1}{x}$ on the interval $[1,b]$. Find the volume that results from rotating R about the x-axis. Call this Vol(b). (See figure on right above.)

b) Now take limit $\lim_{b\to\infty}$ Vol(b). This represents the volume of an 'infinitely' long region which mathematicians call Gabriel's Horn. Do you see why? Is this horn infinite or finite in volume?

c) Suppose the lengths are measured in feet in this problem. Could you fill such a horn with paint? (1 cubic foot = 7.481 gallons.) We will return to this problem later in the term.

Some Additional Answers

8. (a) $\frac{x+2}{2}e^{2x} - \frac{1}{4}e^{2x} + c$; (b) $\frac{1}{2}e^{x^2} + c$; (c) $x\tan x - \ln|secx| + c$

9.

a) $\frac{6^x}{\ln 6} + c$ **b)** $\frac{5^x}{\ln 5} + c$ **c)** $-\frac{3^{\cos(2x)+1}}{2\ln 3} + c$ **d)** $\frac{2^{x^3+3x}}{3\ln 2} + c$

Math 131 Lab 7 Answers

1. $f'(x) = \frac{2}{x} - \frac{1}{8}x \Rightarrow (f'(x))^2 = \frac{4}{x^2} - \frac{1}{2} + \frac{1}{64}x^2$. So

$$AL = \int_1^4 \sqrt{1 + \tfrac{4}{x^2} - \tfrac{1}{2} + \tfrac{1}{64}x^2}\,dx = \int_1^2 \sqrt{\tfrac{4}{x^2} + \tfrac{1}{2} + \tfrac{1}{64}x^2}\,dx = \int_1^4 \sqrt{(\tfrac{2}{x} + \tfrac{1}{8}x^2)^2}\,dx = \int_1^4 \tfrac{2}{x} + \tfrac{1}{8}x^2\,dx$$

$$= 2\ln|x| + \tfrac{1}{16}x^2\Big|_1^4 = (2\ln 4 + 1) - (0 - \tfrac{1}{16}) = 2\ln 4 + \tfrac{15}{16}.$$

2. $f'(x) = \frac{1}{2}x^3 - \frac{1}{2}x^{-3} \Rightarrow (f'(x))^2 = \frac{1}{4}x^6 - \frac{1}{2} + \frac{1}{4}x^{-6}$. So

$$AL = \int_1^2 \sqrt{1 + \tfrac{1}{4}x^6 - \tfrac{1}{2} + \tfrac{1}{4}x^{-6}}\,dx = \int_1^2 \sqrt{\tfrac{1}{4}x^6 + \tfrac{1}{2} + \tfrac{1}{4}x^{-6}}\,dx = \int_1^2 \sqrt{(\tfrac{1}{2}x^3 + \tfrac{1}{2}x^{-3})^2}\,dx = \int_1^2 \tfrac{1}{2}x^3 + \tfrac{1}{2}x^{-3}\,dx$$

$$= \tfrac{1}{8}x^4 - \tfrac{1}{4}x^{-2}\Big|_1^2 = (2 - \tfrac{1}{16}) - (\tfrac{1}{8} - \tfrac{1}{4}) = \tfrac{33}{16}.$$

3. $f'(x) = \frac{1}{2}e^{x/4} - \frac{1}{2}e^{-x/4} \Rightarrow (f'(x))^2 = \frac{1}{4}e^{x/2} - \frac{1}{2} + \frac{1}{4}e^{-x/2}$. So

$$AL = \int_{-1}^1 \sqrt{1 + \tfrac{1}{4}e^{x/2} - \tfrac{1}{2} + \tfrac{1}{4}e^{-x/2}}\,dx = \int_{-1}^1 \sqrt{\tfrac{1}{4}e^{x/2} + \tfrac{1}{2} + \tfrac{1}{4}e^{-x/2}}\,dx = \int_{-1}^1 \sqrt{(\tfrac{1}{2}e^{x/4} + \tfrac{1}{2}e^{-x/4})^2}\,dx$$

$$= \int_{-1}^1 \tfrac{1}{2}e^{x/4} + \tfrac{1}{2}e^{-x/4}\,dx = 2e^{x/4} - 2e^{-x/4}\Big|_1^2 = (2e^{1/4} - 2e^{-1/4}) - (2e^{-1/4} - 2e^{1/4}) = 4(e^{1/4} - e^{-1/4}).$$

4. In work integrals with tanks, D is the density, H represents the height to which the liquid is moved, a and b represent the bottom and top of the liquid to be moved, y represents the height of the layer for which the cross-sectional area $A(y)$ is computed. Since most of these tanks are formed by rotation, the radius will be x, so the cross-sectional area will be $A(y) = \pi x^2$, where we have to solve for x^2 in terms of y.

$$\begin{aligned} W &= D\int_a^b (H-y)\pi A(y)\,dy = 100\int_0^4 (4-y)\pi(4-y)\,dy = 100\pi\int_0^4 (4-y)^2\,dy \\ &= -100\pi[\tfrac{(4-y)^3}{3}]\Big|_0^4 = -100\pi[0 - \tfrac{64}{3}] = 6400\pi/3 \text{ ft} - \text{lbs} \end{aligned}$$

5. The line is $y = 2x - 4 \Rightarrow x = \frac{1}{2}y + 2$. So

$$W = 80\int_0^4 (7-y)\pi(\tfrac{1}{2}y+2)^2\,dy = 80\pi\int_0^4 (7-y)(\tfrac{1}{4}y^2 + 2y + 4)\,dy = 80\pi\int_0^4 -\tfrac{1}{4}y^3 - \tfrac{1}{4}y^2 + 10y + 28\,dy$$

$$= 80\pi\left[-\tfrac{1}{16}y^4 - \tfrac{1}{12}y^3 + 5y^2 + 28y\right]\Big|_0^4 = 80\pi\left[-16 - \frac{16}{3} + 80 + 112\right] = \frac{40960\pi}{3} \text{ ft} - \text{lbs}.$$

6. a) The curves meet at $x = 1$. Note that $y = -1$ is the bottom curve and $y = -\sqrt{x}$ is the top curve. So

$$V = \int_0^1 2\pi x(-\sqrt{x} - (-1))\,dx = \int_0^1 2\pi x(1 - \sqrt{x})\,dx = 2\pi\int_0^1 x - x^{3/2}\,dx 2\pi\left(\frac{x^2}{2} - \frac{2x^{5/2}}{5}\right)\Big|_0^1 = 2\pi\left(\frac{1}{2} - \frac{2}{5}\right) = \frac{\pi}{5}.$$

b) WeBWorK

7. a) Save work! Since the radius of a cross-section of the tank is x, we need to solve for x^2 (not x): but $y = 10 - \frac{1}{2}x^2$, so $x^2 = 20 - 2y$.

$$\begin{aligned} W &= 62.5\int_8^{10} \pi(20-2y)(0-y)\,dy = 62.5\pi\int_8^{10} 2y^2 - 20y\,dy = 62.5\pi[2y^3/3 - 10y^2]\Big|_8^{10} \\ &= 62.5\pi[(2000/3 - 1000) - (1024/3 - 640)] = -6500\pi/3 \text{ ft} - \text{lb} \end{aligned}$$

b) Only the lower limit changes to 9 since the upper part of the barrel leaks out:

$$\begin{aligned} W = 62.5\int_9^{10} \pi(20-2y)(0-y)\,dy &= -62.5\pi[2y^3/3 - 10y^2]\Big|_9^{10} \\ &= -62.5\pi[(2000/3 - 1000) - (486 - 810)] = -1750\pi/3 \text{ ft} - \text{lb} \end{aligned}$$

8. a) Parts:

$u = x+2$ $\quad dv = e^{2x}\,dx$ $du = dx \quad v = \int dv = \frac{1}{2}e^{2x}$	$\int (x+2)e^{2x}\,dx = \frac{x+2}{2}e^{2x} - \int \frac{1}{2}e^{2x}\,dx = \frac{x+2}{2}e^{2x} - \frac{1}{4}e^{2x} + c$

b) Substitution ($u = x^2 \Rightarrow du = 2x\,dx$). $\Rightarrow \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + c = \frac{1}{2}e^{x^2} + c.$

c)

$u = x \quad dv = \sec^2 x\,dx$ $du = dx \quad v = \int dv = \tan x$	$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx = x\tan x - \ln\lvert\sec x\rvert + c$

d)

$u = \ln x \quad dv = x\,dx$ $du = \frac{1}{x}\,dx \quad v = \int dv = \frac{x^2}{2}$	$\int x\ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx = \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$

9. a) $\displaystyle\int 6^x\,dx = \frac{6^x}{\ln 6} + c$

b) $\displaystyle\int 4\cdot 5^x\,dx = \frac{5^x}{\ln 5} + c$

c) $u = \cos(2x)+1,\ du = -2\sin(2x)\,dx.$ $\displaystyle\int 3^{\cos(2x)+1}\sin(2x)\,dx = -\frac{1}{2}\int 3^u\,du = \frac{1}{2}\cdot\frac{3^u}{\ln 3} + c = -\frac{3^{\cos(2x)+1}}{2\ln 3} + c$

d) $u = x^3+3x,\ du = (3x^2+3)\,dx.$ $\displaystyle\int (x^2+1)2^{x^3+3x}\,dx = \frac{1}{3}\int 2^u\,du = \frac{1}{3}\cdot\frac{2^u}{\ln 2} + c = \frac{2^{x^3+3x}}{3\ln 2} + c$

10. You be the professor. Recall this result from Homework: If a and n are a non-zero real numbers, then

$$1 + \left(ax^n - \frac{1}{4a}x^{-n}\right)^2 = \left(ax^n + \frac{1}{4a}x^{-n}\right)^2.$$

a) For your choice of a and n: We need $f(x) = \int ax^n - \frac{1}{4a}x^{-n}\,dx = \frac{a}{n}x^{n+1} + \frac{1}{4a(n-1)}x^{1-n} + c.$

b) Using the homework result above,

$$AL = \int_1^2 \sqrt{1+(f'(x))^2}\,dx = \int_1^2 ax^n + \frac{1}{4a}x^{-n}\,dx = \frac{a}{n}x^{n+1} - \frac{1}{4a(n-1)}x^{1-n}\Big|_1^2$$

$$= \frac{2^{n+1}a}{n+1} - \frac{1}{4a(n-1)2^{n-1}} - \frac{a}{n+1} + \frac{1}{4a(n-1)}.$$

11. Cross-sectional area: $A(x) = \frac{1}{2}bh = \frac{1}{2}(2\sqrt{x})(2\sqrt{x}) = 2x$. So

$$V = \int_0^9 A(x)\,dx = \int_0^9 2x\,dx = x^2\Big|_0^9 = 81\ \text{cm}^3$$

12. a) $\text{Vol}(b) = \pi\int_1^b (x^{-1})^2\,dx = \pi\int_1^b x^{-2}\,dx = -\pi(\frac{1}{x})\Big|_1^b = -\pi(\frac{1}{b} - 1)$ cu-ft.

b) Using the integration above: $\lim_{b\to\infty}\text{Vol}(b) = \lim_{b\to\infty}\pi(-\frac{1}{b}+1) = \pi$ cu-ft.

c) It would require $\pi 7.481 = 23.502$ gallons.