Math 131 Lab 8

☞ **Do * problems first. Then do the others.** Brief answers are on the back. Reduction formulas on back

1. Higher Powers of Trig Functions Use the rules we developed in class to determine

a) $*\int \sin^2 2x \cos^3 2x\,dx$ **b)** $*\int \cos^5 2x\,dx$ **c)** $\int \sec^5(2x)\,dx$

d) $*\int \sin^3(5x)\cos^{-7}(5x)\,dx$ **e)** $\int \cos^2(10x)\sin^2(10x)\,dx$ **f)** $\int \tan^2(\pi x)\sec^2(\pi x)\,dx$ Think first!

2. Low Powers of Trig Functions which you should already know.

a) $\int \cos 9x\,dx$ **b)** $*\int \tan^2(\pi x)\,dx$ **c)** $*\int \sin^2(-7x)\,dx$

3. Before working these out, classify each by the technique that applies: mentally adjust, substitution, parts, parts twice, or ordinary methods. Check your answers by differentiating.

a) $*\int xe^{x^2}\,dx$ **b)** $*\int_0^1 (x^2-x+1)e^x\,dx$ **c)** $*\int_0^\pi (x-\pi)\sin x\,dx$

d) $\int e^x \sin(e^x)\,dx$ **e)** $\int e^{3x}\sin x\,dx$ **f)** $*\int \sin(5x)\cos(x)\,dx$

4. a) *Assume that $n \neq -1$. Determine a formula for $\int x^n \ln x\,dx$.

b) Use your answer to quickly determine (without further integration) $\int \frac{\ln x}{x^6}\,dx$.

5. a) *Determine $\int (\ln x)^2\,dx$.

b) *Let R be the region enclosed by $y = \ln x$, the x-axis, and $x = e$ in the first quadrant. Rotate R about the x-axis and find the volume. Use your answer to part (a).

6. Area Review: Find the area of the region enclosed by $y = \arcsin x$, the x-axis, and $x = \sqrt{2}/2$ in the first quadrant by integrating along the x-axis.

7. a) ***Work Review:** A tank is formed from rotating the region enclosed by $y = \arcsin(x^2)$, the y-axis, $y = \pi/2$, and the x-axis. The tank is full of whale oil (60 lbs/ft^3). Find the work "lost" (the work will be negative) by draining the tank through a hole in the bottom. Hint: Remember H represents the height where the liquid ends up. What is H in this problem? (Set up the integral first and decide on a technique. Solve later.)

b) Just set up the integral for the work is lost if only the top half is drained.

8. ***Volume Review:** Let R be the region in the first quadrant enclosed by $y = \cos x$, $y = \sin x$, and the y-axis.

a) Rotate R about the x-axis and find the volume using the disk method. (Do set-up now, check answer later.)

b) Rotate R about the y-axis and find the volume by using the shell method. (Do set-up now, check answer later.)

9. WeBWorK `setDay21`. Let R be the region enclosed by $y = \ln x$, the y-axis, the x-axis, and $y = 1$ in the first quadrant. Rotate R about the y-axis to form a tank. If liquid has density 60 lbs/cu. ft., how much work is required to fill it through a hole in the bottom of the tank? Hint: During the integration you can use Problem 4.

10. [Harder] Let R be the region enclosed by $y = e^x$, the y-axis, and $y = e$ in the first quadrant. Rotate R about the y-axis to form a tank. If it is full of a liquid whose density is 60 lbs/cu. ft., how much work is lost if it leaks out the bottom and drops to ground level?

11. a) In the past we have done the problem $\int \frac{\ln x}{x}\,dx$ by u-substitution. Show that **you can do it by parts**—but as a problem that cycles back on itself (a technique that we had not seen until Day 20–21).

b) Check your answer by doing the same problem via substitution.

12. a) WeBWorK `setExtraCredit2`. Assume that n is a positive integer. Create a reduction formula for $\int x^n e^x\,dx$ by using integration by parts once. Hint: Let $u = x^n$.

b) Extra Credit: Use your reduction formula (repeatedly) to find $\int x^3 e^x\,dx$.

Math 131 Lab 8 Brief Answers. Caution: Be careful of typos. **Extra Credit** if you are the first to find one.

1. a) $\dfrac{\sin^3 2x}{6} - \dfrac{\sin^5 2x}{10} + c$

b) $\dfrac{\cos^4 2x \sin 2x}{10} + \dfrac{2\cos^2 2x \sin 2x}{15} + \dfrac{4 \sin 2x}{15} + c$

c) $\dfrac{\sec^3(2x)\tan(2x)}{8} + \dfrac{3\sec(2x)\tan(2x)}{16} + \dfrac{3\ln|\sec(2x)+\tan(2x)|}{16} + c$

d) $\dfrac{\cos^{-6}(5x)}{30} - \dfrac{\cos^{-4}(5x)}{20} + c$

e) $\frac{1}{3\pi}\tan^3(\pi x) + c$

2. Use half-angle for (c).

a) $\frac{1}{9}\sin 9x + c$ **b)** $\frac{1}{\pi}\tan(\pi x) - x + c$ **c)** $\dfrac{x}{2} + \dfrac{1}{28}\sin(-14x) + c$

3. Sketches:

a) $\frac{1}{2}e^{x^2} + c$ **b)** $(x^2 - 3x + 4)e^x\Big|_0^1 = 2e - 4$ **c)** $-(x-\pi)\cos x + \sin x\Big|_0^\pi = -\pi$

d) $-\cos e^x + c$ **e)** $\frac{1}{10}(3e^{3x}\sin x - e^{3x}\cos x) + c$ **f)** $-\dfrac{1}{24}[\sin(5x)\sin x + 5\cos(5x)\cos x] + c$

4. a) $\frac{1}{n+1}x^{n+1}\ln x - \frac{1}{(n+1)^2}x^{n+1} + c$; **(b)** $-\frac{1}{5}x^{-5}\ln(x) - \frac{1}{25}x^{-5} + c$

5. a) Use parts twice: $x(\ln x)^2 - 2x\ln x + 2x + c$.

b) Use (a): $\pi(e-2)$.

6. By parts: $x \arcsin x + \sqrt{1-x^2}\Big|_0^{\sqrt{2}/2} = \frac{\sqrt{2}}{8}\pi + \sqrt{2}/2 - 1$.

7. a) $W = 60\int_0^{\pi/2} \pi(0-y)(\sin y)\,dy = -60\pi\int_0^{\pi/2} y\sin y\,dy$. Use parts: -60π ft-lbs. (b) $-60\pi\int_{\pi/4}^{\pi/2} y \sin y\,dy$.

8. a) $V = \pi\int_0^{\pi/4}\cos^2 x - \sin^2 x\,dx = \frac{1}{2}\pi$. **(b)** $V = 2\pi\int_0^{\pi/4} x\cos x - x\sin x\,dx = \frac{\sqrt{2}}{2}\pi^2 - 2\pi$.

9. $-15\pi(e^2+1)$ ft-lbs.

10. $-15\pi(e^2-1)$ ft-lbs.

11. $\frac{(\ln x)^2}{2} + c$.

12. a) $x^n e^x - n\int x^{n-1}e^x\,dx$. **(b)** $x^3e^x - 3x^2e^x + 6xe^x - 6e^x + c$.

Reduction Formulas for Large Powers.

1) $\displaystyle\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int\cos^{n-2}x\,dx$

2) $\displaystyle\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int\sin^{n-2}x\,dx$

3) $\displaystyle\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1}x - \int\tan^{n-2}x\,dx$

4) $\displaystyle\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2}x\tan x + \frac{n-2}{n-1}\int\sec^{n-2}x\,dx$

Math 131 Lab 8 Answers

1. a) Since the power of the cosine function is odd, we use Guideline #2: Split off a power of $\cos 2x$, then let $u = \sin 2x$, so $\frac{1}{2}du = \cos 2x\,dx$.

$$\int \sin^2 2x \cos^3 2x\,dx = \int \sin^2 2x \cos^2 2x \cdot \cos 2x\,dx = \int \sin^2 2x(1-\sin^2 2x)\cdot \cos 2x\,dx$$

$$= \frac{1}{2}\int u^2(1-u^2)\,du = \frac{1}{2}\int u^2 - u^4\,du = \frac{1}{2}\left[\frac{u^4}{3} - \frac{u^5}{5}\right] + c = \frac{\sin^3 2x}{6} - \frac{\sin^5 2x}{10} + c$$

b) Use reduction formulas. First let $u = 3x$, so $\frac{1}{3}du = dx$. So $\displaystyle\int \cos^4 3x\,dx = \frac{1}{3}\int \cos^4 u\,du$

$$= \frac{1}{3}\left(\frac{\cos^3 u \sin u}{4} + \frac{3}{4}\int \cos^2 u\,du\right) = \frac{1}{3}\left(\frac{\cos^3 u\sin u}{4} + \frac{3}{4}\left[\frac{\cos u \sin u}{2} + \frac{1}{2}\int 1\,du\right]\right)$$

$$= \frac{1}{3}\left(\frac{\cos^3 u \sin u}{4} + \frac{3}{4}\left[\frac{\cos u\sin u}{2} + \frac{u}{2}\right]\right) + c = \frac{\cos^3 3x \sin 3x}{12} + \frac{\cos 3x \sin 3x}{8} + \frac{3x}{8} + c$$

c) Use reduction formula and substitution: $u = 2x \Rightarrow \frac{1}{2}du = dx$. So

$$\int \sec^5(2x)\,dx = \frac{1}{2}\int \sec^5 u\,du = \frac{1}{2}\left[\frac{\sec^3 u \tan u}{4} + \frac{3}{4}\int \sec^3 u\,du\right] = \frac{1}{2}\left[\frac{\sec^3 u\tan u}{4} + \frac{3}{4}\left[\frac{\sec u \tan u}{2} + \frac{1}{2}\int \sec\,du\right]\right]$$

$$= \frac{1}{2}\left[\frac{\sec^3 u \tan u}{4} + \frac{3\sec u \tan u}{8} + \frac{3\ln|\sec u + \tan u|}{8}\right] + c = \frac{\sec^3(2x)\tan(2x)}{8} + \frac{3\sec(2x)\tan(2x)}{16} + \frac{3\ln|\sec(2x)+\tan(2x)|}{16} + c$$

d) Since the power of the sine function is odd, we use Guideline #1: Split off a power of $\sin 5x$, then let $u = \cos 5x$, so Use $u = \cos(5x)$ so $-\frac{1}{5}du = \sin(5x)dx$.

$$\int \sin^3(5x)\cos^{-7}(5x)\,dx = \int \sin^2(5x)\cos^{-7}(5x)\sin(5x)\,dx = \int [1-\cos^2(5x)]\cos^{-7}(5x)\sin(5x)\,dx$$

$$= -\frac{1}{5}\int (1-u^2)u^{-7}\,du = -\frac{1}{5}\int u^{-7} - u^{-5})\,du = \frac{u^{-6}}{30} - \frac{u^{-4}}{20} + c = \frac{\cos^{-6}(5x)}{30} - \frac{\cos^{-4}(5x)}{20} + c$$

e) Use Half-angle forumlas:

$$\int \cos^2(10x)\sin^2(10x)\,dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos(20x)\right)\left(\frac{1}{2} - \frac{1}{2}\cos(20x)\right)\,dx = \int \frac{1}{4} - \frac{1}{4}\cos^2(20x)\,dx$$

$$= \int \frac{1}{4} - \frac{1}{4}\left[\frac{1}{2} + \frac{1}{2}\cos(40x)\right]dx = \int \frac{1}{8} - \frac{1}{8}\cos(40x)\,dx = \frac{x}{8} - \frac{1}{320}\sin(40x) + c$$

f) u-substitution.

Substitution:	$u = \tan \pi x$	$= \frac{1}{\pi}\int u^2\,du = \frac{1}{3\pi}u^3 + c = \frac{1}{3\pi}\tan^3(\pi x) + c$
$du = \pi \sec^2 \pi x\,dx$	$\frac{1}{\pi}du = x\,dx$	

2. a) Mental adjustment $(u = 9x)$.$\int \cos 9x\,dx = \frac{1}{9}\int \cos u\,du = \frac{1}{9}\sin u + c = \frac{1}{9}\sin 9x + c$.

b) Trig id: $\int \tan^2(\pi x)\,dx = \int \sec^2(\pi x) - 1\,dx = \frac{1}{\pi}\tan(\pi x) - x + c$.

c) Use half-angle formula and followed by a mental adjustment:

$$\int \sin^2(7x)\,dx = \int \frac{1}{2} - \frac{1}{2}\cos(-14x)\,dx = \frac{1}{2}x + \frac{1}{28}\sin(-14x) + c$$

3. a) Substitution ($u = x^2 \Rightarrow du = 2x\,dx$). $\Rightarrow \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + c = \frac{1}{2}e^{x^2} + c$.

b) Parts twice:

$u = x^2 - x + 1$ $du = (2x-1)dx$	$dv = e^x dx$ $v = e^x$	$= (x^2 - x + 1)e^x\Big\vert_0^1 - \int_0^1 (2x-1)e^x\,dx$
$u = 2x - 1$ $du = 2dx$	$dv = e^x dx$ $v = e^x$	$= (x^2 - x + 1)e^x\Big\vert_0^1 - [(2x-1)e^x\Big\vert_0^1 - \int_0^1 2e^x\,dx]$ $= (x^2 - x + 1)e^x - (2x-1)e^x + 2e^x\Big\vert_0^1 = 2e - 4$

c) Parts:

$u = x - \pi$ $du = dx$	$dv = \sin x dx$ $v = -\cos x$	$= -(x-\pi)\cos x\Big\vert_0^\pi + \int_0^\pi \cos x\,dx$ $= -(x-\pi)\cos x + \sin x\Big\vert_0^\pi = (0+0) - (\pi) = -\pi$

d) Substitution ($u = e^x \Rightarrow du = e^x\,dx$). $\Rightarrow \int \sin u\,du = -\cos u + c = -\cos(e^x) + c$.

e) Parts twice, "circle around". Note signs and constants:

$u = e^{3x}$ $du = 3e^{3x}dx$	$dv = \sin x dx$ $v = -\cos x$	$\int e^{3x}\sin x\,dx = -e^{3x}\cos x + \int 3e^{3x}\cos x\,dx$
$u = 3e^{3x}$ $du = 9e^{3x}dx$	$dv = \cos x dx$ $v = \sin x$	$\int e^{3x}\sin x\,dx = -e^{3x}\cos x + 3e^{3x}\sin x - 9\int e^{3x}\sin x\,dx$ So, $10\int e^{3x}\sin x\,dx = e^{3x}(3\sin x - \cos x) + c$ Thus, $\int e^{3x}\sin x\,dx = \frac{1}{10}e^{3x}(3\sin x - \cos x) + c$

f) Parts twice, "circle around". Note signs and constants:

$u = \sin(5x)$ $du = 5\cos(5x)\,dx$	$dv = \cos x\,dx$ $v = \sin x$	$\int \sin(5x)\cos x\,dx = \sin(5x)\sin x - \int 5\cos(5x)\sin x\,dx$
$u = 5\cos(5x)$ $du = -25\sin(5x)\,dx$	$dv = \sin x\,dx$ $v = -\cos x$	$\int \sin(5x)\cos x\,dx = \sin(5x)\sin x + 5\cos(5x)\cos x + \int 25\sin(5x)\cos x\,dx$ So, $-24\int \sin(5x)\cos x\,dx = \sin(5x)\sin x + 5\cos(5x)\cos x + c$ Thus, $\int \sin(5x)\cos x\,dx = -\frac{1}{24}[\sin(5x)\sin x + 5\cos(5x)\cos x] + c$

4. a) Parts:

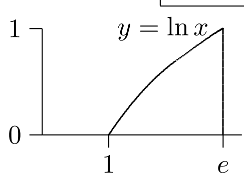
$u = \ln x$ $du = x^{-1}dx$	$dv = x^n\,dx$ $v = \frac{1}{n+1}x^{n+1}$	$\int x^n \ln x\,dx = \frac{1}{n+1}x^{n+1}\ln x - \int \frac{1}{n+1}x^n\,dx$ $\int x^n \ln x\,dx = \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{(n+1)^2}x^{n+1} + c$

b) $-\frac{1}{5}x^{-5}(\ln(x) + \frac{1}{5}) + c$.

5. a) Parts twice:

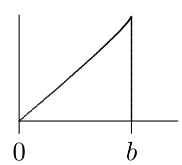
$u = (\ln x)^2$ $du = \frac{2\ln x}{x}dx$	$dv = dx$ $v = x$	$= x(\ln x)^2 - 2\int \ln x\,dx$. Use parts again in problem #4 with $n = 0$. $= x(\ln x)^2 - 2x\ln x + 2x + c$

b)



From (a): $\int_1^e \pi(\ln x)^2\,dx = \pi\left[x(\ln x)^2 - 2x\ln x + 2x\right]\Big\vert_1^e = \pi(e-2)$.

6. $A = \int_0^{b=\sqrt{2}/2} \arcsin x\,dx$



Parts:

$u = \arcsin x$ $du = \frac{1}{\sqrt{1-x^2}}\,dx$	$dv = dx$ $v = x$	$= x\arcsin x\Big\vert_0^{\sqrt{2}/2} - \int_0^{\sqrt{2}/2} \frac{x}{\sqrt{1-x^2}}\,dx$
Substitution: $du = -2x\,dx$	$u = 1 - x^2$ $-\frac{1}{2}du = x\,dx$	$= x\arcsin x\Big\vert_0^{\sqrt{2}/2} + \frac{1}{2}\int u^{-1/2}\,du = x\arcsin x\Big\vert_0^{\sqrt{2}/2} + u^{1/2}$ $= x\arcsin x + \sqrt{1-x^2}\Big\vert_0^{\sqrt{2}/2} = \frac{\sqrt{2}}{8}\pi + \frac{\sqrt{2}}{2} - 1$

7. a) Save work! Since the radius of a cross-section of the tank is x, we need to solve for x^2 (not x): $y = \arcsin(x^2)$, so $x^2 = \sin y$. $W = 60\displaystyle\int_0^{\pi/2} \pi(0-y)(\sin y)\,dy = -60\pi\int_0^{\pi/2} y\sin y\,dy$. Use parts:

$u = y$	$dv = \sin y\,dy$	$= -60\pi\left[-y\cos y\Big\vert_0^{\pi/2} + \int_0^{\pi/2}\cos y\,dy\right]$
$du = dy$	$v = -\cos y$	$= -60\pi\left[(0-0)+\sin y\Big\vert_0^{\pi/2}\right] = (0+0)+(1-0)) = -60\pi$ lbf

b) Only the lower limit changes to $\pi/4$ since the upper part of the tank drains out: $W = 60\displaystyle\int_{\pi/4}^{\pi/2} \pi(0-y)(\sin y)\,dy$

8. a) $V = \pi\int_0^{\pi/4}\cos^2 x - \sin^2 x\,dx = \pi\int_0^{\pi/4}(\frac{1}{2}+\frac{1}{2}\cos 2x) - (\frac{1}{2}-\frac{1}{2}\cos 2x)\,dx = \pi\int_0^{\pi/4}\cos 2x)\,dx = \frac{1}{2}\pi\sin(2x)\Big|_0^{\pi/4} = \frac{1}{2}\pi$.

b) $V = 2\pi\int_0^{\pi/4} x\cos x - x\sin x\,dx = 2\pi\int_0^{\pi/4} x(\cos x - \sin x)\,dx$. Use parts

$u = x$ $du = dx$	$dv = (\cos x - \sin x)dx$ $v = \sin x + \cos x$	$2\pi\int_0^{\pi/4} x(\cos x - \sin x)\,dx = 2\pi\left[x(\sin x + \cos x) - \int(\sin x + \cos x)\,dx\right]_0^{\pi/4}$
		$2\pi\int_0^{\pi/4} x\cos x - x\sin x\,dx = 2\pi\left[x(\sin x + \cos x) + \cos x - \sin x\,dx\right]_0^{\pi/4} = \frac{\sqrt{2}}{2}\pi^2 - 2\pi$

So $\pi\int_1^{e^2}(\ln x)^2\,dx = \pi[x(\ln x)^2 - 2x\ln x + 2x]\Big|_1^{e^2} = \pi[(4e^2 - 4e^2 + 2e^2) - (0 - 0 + 1) = \pi(2e^2 - 1)$.

9. Since $y = \ln x$, then $x = e^y$. So $W = \int_0^1 60\pi(e^y)^2(y-0)\,dy = 60\pi\int_0^1 ye^{2y}\,dy$. By parts

$u = y$ $du = dy$	$dv = e^{2y}\,dy$ $v = \frac{1}{2}e^{2y}$	$= 60\pi[\frac{1}{2}ye^{2y}\Big\vert_0^1 - \int_0^1 \frac{1}{2}e^{2y}\,dy]$ $=60\pi[\frac{1}{2}ye^{2y} - \frac{1}{4}e^{2y}]\Big\vert_0^1$
		$-60\pi[(\frac{1}{2}e^2 - \frac{1}{4}e^2) - (0 - \frac{1}{4})] = -60\pi[\frac{1}{4}e^2 + \frac{1}{4}] = -15\pi(e^2+1)$

10. Since $y = e^x$, then $x = \ln y$. So $W = \int_1^e 60\pi(\ln y)^2(0-y)\,dy = -60\pi\int_1^e y(\ln y)^2\,dy$. By parts twice, the second time using #4 with $n = 1$:

$u = (\ln y)^2$ $du = \frac{2\ln y}{y}\,dy$	$dv = y\,dy$ $v = \frac{1}{2}y^2$	$= -60\pi[y(\ln y)^2\Big\vert_1^e - \int_1^e y\ln y\,dy]$
Use #4 with $n = 1$:		$= -60\pi[\frac{1}{2}y^2(\ln y)^2 - \frac{1}{2}y^2\ln y + \frac{1}{4}y^2]\Big\vert_1^e$ $=-60\pi[(\frac{1}{2}e^2 - \frac{1}{2}e^2 + \frac{1}{4}e^2) - (0 + 0 + \frac{1}{4})] = -15\pi(e^2-1)$

11. a) Parts:

$u = \ln x$ $du = \frac{1}{x}\,dx$	$dv = \frac{1}{x}\,dx$	$\int\frac{\ln x}{x}\,dx = (\ln x)^2 - \int\frac{\ln x}{x}\,dx$ Cycle! So $2\int\frac{\ln x}{x}\,dx = (\ln x)^2$ or $\int\frac{\ln x}{x}\,dx = \frac{(\ln x)^2}{2} + c$

b) EZ u-substitution: $u = \ln x$, $du = \frac{1}{x}\,dx$. So $\displaystyle\int\frac{\ln x}{x}\,dx = \int u\,du = \frac{u^2}{2} + c = \frac{(\ln x)^2}{2} + c$

12. a) Parts:

$u = x^n$ $du = nx^{n-1}dx$	$dv = e^x\,dx$ $v = e^x$	$\int x^ne^x\,dx = x^ne^x - n\int x^{n-1}e^x\,dx$

b) $\displaystyle\int x^3e^x\,dx = x^3e^x - 3\int x^2e^x\,dx = x^3e^x - 3\left[x^2e^x - 2\int xe^x\,dx\right]$

$$= x^3e^x - 3\left[x^2e^x - 2(xe^x - \int e^x\,dx)\right] = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + c.$$