Math 131 Lab 10

1. Try integrating these rational functions. One of them is NOT by partial fractions.

a) $\displaystyle\int \frac{4-x}{x^3+x^2-2x}\,dx$ **b)** $\displaystyle\int \frac{6x^2-8}{x^3-4x}\,dx$ **c)** $\displaystyle\int \frac{2x}{(x-1)^3}\,dx$

2. Use #1(c) to evaluate the improper integral $\displaystyle\int_3^\infty \frac{2x}{(x-1)^3}\,dx$. You have already done the anti-differentiation, so just evaluate the appropriate limit. Use proper notation!

3. OK, here's an easy one. The p-power theorem will be very important. Use it to determine these improper integrals. Do not do any integration. Just use the theorem.

a) $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ **b)** $\displaystyle\int_1^\infty \frac{1}{x^{1/2}}\,dx$ **c)** $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ **d)** $\displaystyle\int_1^\infty \frac{1}{x^{10}}\,dx$ **e)** $\displaystyle\int_1^\infty \frac{1}{x^{4/3}}\,dx$

4. Evaluate each improper integral; determine whether it converges or not. **Use proper notation! Suggestion:** Do the indefinite integral first and then use your work to evaluate the improper integral, as you did in problem #2.

a) $\displaystyle\int_2^\infty \frac{2}{x^2-1}\,dx$ **b)** $\displaystyle\int_{-\infty}^0 x^2e^{x^3}\,dx$ **c)** $\displaystyle\int_0^\infty \frac{2x}{\sqrt[3]{1+x^2}}\,dx$ triangle or not?

Come back to the next two after doing some limits below.

d) $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx$ **e)** $\displaystyle\int_0^\infty \frac{1}{(1+x^2)^{3/2}}\,dx$

5. Some interesting limits. Classify each as indeterminate (indicate what type: "$\frac{0}{0}$", "$\frac{\infty}{\infty}$", "$0\cdot\infty$", "1^∞", "∞^0", "0^0", or "$\infty-\infty$") or **not** indeterminate. Answers not in order: 0, 0, 0, 0, $\ln 4$, 1, 2, e^2, e^k, $+\infty$, and -6.

a) $\displaystyle\lim_{x\to 0}\frac{\cos 4x-\cos 2x}{x^2}$ **b)** $\displaystyle\lim_{x\to 0^+} 2x\ln x$

c) $\displaystyle\lim_{x\to\infty} x^2e^{-x}$ **d)** $\displaystyle\lim_{x\to 0^+}\frac{\cos x}{x^2}$ **e)** $\displaystyle\lim_{x\to\infty}[\ln(4x+9)-\ln(x+7)]$ **f)** $\displaystyle\lim_{x\to\infty}\frac{x\ln x}{x^2+1}$

g) $\displaystyle\lim_{x\to 0^+}(2x)^x$ **h)** $\displaystyle\lim_{x\to 0^+}(1+2x)^{1/x}$ **i)** $\displaystyle\lim_{x\to\infty}\left(1+\tfrac{k}{x}\right)^x$ **j)** $\displaystyle\lim_{x\to\infty}\left(1+\tfrac{1}{x}\right)^{x^2}$

k) $\displaystyle\lim_{x\to 0}\frac{\arctan 4x}{\sin 2x}$ **l)** $\displaystyle\lim_{x\to 0}\frac{\sin 4x}{3\sec x}$

6. Harder Evaluation: Evaluate the improper integral $\displaystyle\int_2^\infty \frac{4-x}{x^3+x^2-2x}\,dx$. You have already done the anti-differentiation in #1(a), so just evaluate the appropriate limit. You will need to use log properties to simplify the limit calculation. Use proper notation!

Do these only after completing all of the problems above. Practice for the next Exam.

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7. Try this octet of problems requiring a variety of techniques. Begin by determining which techniques you will use for each.

a) $\displaystyle\int \frac{x}{\sqrt{4-x}}\,dx$ **b)** $\displaystyle\int \frac{x}{\sqrt{4-x^2}}\,dx$ **c)** $\displaystyle\int \frac{4}{4-x^2}\,dx$ **d)** $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx$

e) $\displaystyle\int \frac{4}{4+x^2}\,dx$ **f)** $\displaystyle\int \frac{4x}{4-x^2}\,dx$ **g)** $\displaystyle\int \frac{1}{x^2\sqrt{x^2-4}}\,dx$ **h)** $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$

Some Answers

Complete detailed answers are online. Caution: Watch for typos. First to find a particular mistake gets Extra Credit.

1.

a) $-2\ln|x| + \ln|x-1| + \ln|x+2| + c$

b) $2\ln|u| = 2\ln|x^3 - 4x| + c$

c) $-2(x-1)^{-1} - (x-1)^{-2} + c$

2. 5/4.

3. 1, diverges, diverges, $\frac{1}{9}$, and 3.

4. a) $\lim_{b\to\infty} \ln\frac{b-1}{b+1} - \ln\frac{1}{3} = \ln 1 + \ln 3 = \ln 3$

b) $\lim_{a\to-\infty} \frac{e^0}{3} - \frac{e^{a^3}}{3} \overset{a^3\to-\infty}{=} \frac{1}{3} - \frac{0}{3} = \frac{1}{3}$.

c) $\lim_{b\to\infty} \frac{3}{2}(1+b^2)^{2/3} - \frac{3}{2} = +\infty$. Diverges

d) $\lim_{b\to\infty} \frac{1}{2}(\ln x)^2\Big|_1^\infty = \lim_{b\to\infty} \frac{1}{2}[(\ln b)^2 - 0] = \infty$ Diverges.

e) 1

6. 0

7. Use a variety of techniques.

a) $-8\sqrt{4-x} + \dfrac{2}{3}(4-x)^{3/2} + c$

b) $-\sqrt{4-x^2} + c$

c) $\ln|2+x| - \ln|2-x| + c$

d) $\arcsin\dfrac{x}{2} + c$

e) $2\arctan\dfrac{x}{2} + c$

f) $-2\ln|4-x^2| + c$

g) $\dfrac{\sqrt{x^2-4}}{4x} + c$

h) $2\arcsin(\frac{x}{2}) - \frac{1}{2}x\sqrt{4-x^2} + c$

Math 131 Lab 10: Answers

1. Use partial fractions for all but part (b):

a) Homework

b) Substitution: $u = x^3 - 4x$, $du = (3x^2 - 4)\,dx$. So $2du = (6x^2 - 8)\,dx$. $= 2\int \frac{1}{u}\,du = 2\ln|u| = 2\ln|x^3 - 4x| + c$.

c) $$\frac{2x}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} = \frac{Ax^2 - 2Ax + A + Bx - B + C}{(x-1)^3}.$$

$$\begin{array}{llllll} x^2: & A & & & = 0 \\ x: & -2A+ & B & & = 2 \\ \text{const:} & A & -B+ & C & = 0 \end{array} \Rightarrow A = 0,\ B = 2 = C.\ \int \frac{2}{(x-1)^2} + \frac{2}{(x-1)^3}\,dx = -2(x-1)^{-1} - (x-1)^{-2} + c.$$

2. a) $$\int_3^\infty \frac{2x}{(x-1)^3}\,dx = \lim_{b\to\infty} \int_3^\infty \frac{2x}{(x-1)^3}\,dx = \lim_{b\to\infty} -2(x-1)^{-1} - (x-1)^{-2}\Big|_3^\infty$$

$$= \lim_{b\to\infty} -\frac{2}{(b-1)} - \frac{1}{(b-1)^2} - \left(-1 - \frac{1}{4}\right) = -0 - 0 + \frac{5}{4} = \frac{5}{4}$$

3. $$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1}, & \text{if } p > 1 \\ \text{Diverges} & \text{if } p \le 1 \end{cases}$$. So the answers are: 1, diverges, diverges, $\frac{1}{9}$, and 3.

4. In each I first outline the antiderivative, then do the improper integral.

a) Partial fractions: $\int \frac{2}{x^2-1}\,dx = \int \frac{1}{x-1} - \frac{1}{x+1}\,dx = \ln|x-1| - \ln|x+1| = \ln\left|\frac{x-1}{x+1}\right|$. So

$$\int_2^\infty \frac{2}{x^2-1}\,dx = \lim_{b\to\infty} \int_2^b \frac{2}{x^2-1}\,dx = \lim_{b\to\infty} \ln\left|\frac{x-1}{x+1}\right|\Big|_2^b = \lim_{b\to\infty} \ln\left|\frac{b-1}{b+1}\right| - \ln\frac{1}{3} = \ln 1 + \ln 3 = \ln 3.$$

b) u-sub: $u = x^3$, $du = 3x^2\,dx$, and $\frac{1}{3}\,du = x^2\,dx$. So $\displaystyle\int x^2 e^{x^3}\,dx = \frac{1}{3}\int e^u\,du = \frac{e^u}{3} + c = \frac{e^{x^3}}{3} + c$. So

$$\int_{-\infty}^0 x^2 e^{x^3}\,dx = \lim_{a\to-\infty} \int_a^0 x^2 e^{x^3}\,dx = \lim_{a\to-\infty} \frac{e^{x^3}}{3}\Big|_a^0 = \lim_{a\to-\infty} \frac{e^0}{3} - \frac{e^{-a^3}}{3} \overset{a^3\to-\infty}{=} \frac{1}{3} - \frac{0}{3} = \frac{1}{3}.$$

c) u-substitution ($u = 1 + x^2$): $\int \frac{2x}{\sqrt[3]{1+x^2}}\,dx = \frac{3}{2}(1+x^2)^{2/3}$. So

$$\int_0^\infty \frac{2x}{\sqrt[3]{1+x^2}}\,dx = \lim_{b\to\infty} \int_0^b \frac{2x}{\sqrt[3]{1+x^2}}\,dx = \lim_{b\to\infty} \frac{3(1+x^2)^{2/3}}{2}\Big|_0^b = \lim_{b\to\infty} \frac{3(1+b^2)^{2/3}}{2} - \frac{3}{2} = \infty \text{ Diverges.}$$

d) u-substitution: $u = \ln x$. Then $\int \frac{\ln x}{x}\,dx = \int u\,du = \frac{1}{2}u^2 + c = \frac{1}{2}(\ln x)^2 + c$. So

$$\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{b\to\infty} \int_1^\infty \frac{\ln x}{x}\,dx = \lim_{b\to\infty} \frac{1}{2}(\ln x)^2\Big|_1^\infty = \lim_{b\to\infty} \frac{1}{2}[(\ln b)^2 - 0] = \infty \text{ Diverges.}$$

e) Homework

5. Make sure to check those stages at which l'Hôpital's rule applies.

a) $\frac{0}{0}$. $$\lim_{x\to0} \frac{\cos 4x - \cos 2x}{x^2} \overset{\text{l'Ho}}{=} \lim_{x\to0} \frac{-4\sin 4x + 2\sin 2x}{2x} \overset{\text{l'Ho}}{=} \lim_{x\to0} \frac{-16\cos 4x + 4\cos 2x}{2} = \frac{-16+4}{2} = -6.$$

b) $0\cdot\infty$. $$\lim_{x\to0^+} 2x\ln x = \lim_{x\to0^+} \frac{2\ln x}{\frac{1}{x}} \overset{\text{l'Ho}}{=} \lim_{x\to0^+} \frac{\frac{2}{x}}{-\frac{1}{x^2}} = \lim_{x\to0^+} -\frac{2x^2}{x} = \lim_{x\to0^+} -2x = 0.$$

c) $\infty\cdot0$. $$\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

d) Not indeterminate. $\displaystyle\lim_{x\to0^+} \frac{\cos x}{x^2} \to \frac{1}{0^+}$: $+\infty$. l'Hôpital's rule does not apply.

e) $\infty - \infty$. $$\lim_{x\to\infty} \ln(4x+9) - \ln(x+7) = \lim_{x\to\infty} \ln\left(\frac{4x+9}{x+7}\right) = \lim_{x\to\infty} \ln\left(\frac{4+\frac{9}{x}}{1+\frac{7}{x}}\right) = \ln 4.$$

f) $\frac{\infty}{\infty}$. $\displaystyle\lim_{x\to\infty}\frac{x\ln x}{x^2+1}\overset{\text{l'Ho}}{=}\lim_{x\to\infty}\frac{1+\ln x}{2x}\overset{\text{l'Ho}}{=}\lim_{x\to\infty}\frac{\frac{1}{x}}{2}=\frac{0}{2}=0.$

g) 0^0. Let $y=\displaystyle\lim_{x\to0^+}(3x)^x$. We want to find y. Using the log process,

$$\begin{aligned}\ln y=\ln(\lim_{x\to0^+}(3x)^x)\Longrightarrow \ln y &\overset{\text{Cont}}{=} \lim_{x\to0^+}\ln(3x)^x\\ \ln y &= \lim_{x\to0^+}x\ln 3x\\ \ln y &= \lim_{x\to0^+}\frac{\ln 3x}{\frac{1}{x}}\\ \ln y &\overset{\text{l'Ho}}{=} \lim_{x\to0^+}\frac{\frac{3}{3x}}{-\frac{1}{x^2}}\\ \ln y &= \lim_{x\to0^+}-x=0.\end{aligned}$$

But $\ln y=0$ implies $y=e^0=1$. So $\displaystyle\lim_{x\to0^+}(3x)^x=y=1$.

h) 1^∞. Let $y=\displaystyle\lim_{x\to0^+}(1+2x)^{1/x}$, so

$$\begin{aligned}\ln y &= \ln\lim_{x\to0^+}(1+2x)^{1/x}\\ \ln y &\overset{\text{Cont}}{=} \lim_{x\to0^+}\ln(1+2x)^{1/x}\\ \ln y &= \lim_{x\to0^+}\frac{1}{x}\ln(1+2x)\\ \ln y &= \lim_{x\to0^+}\frac{\ln(1+2x)}{x}\\ \ln y &\overset{\text{l'Ho}}{=} \lim_{x\to0^+}\frac{\frac{2}{1+2x}}{1}=\frac{2}{1}=2.\end{aligned}$$

But $\ln y=2$ implies $y=e^2$. So $\displaystyle\lim_{x\to0^+}(1+2x)^{1/x}=e^2$.

i) $\infty-\infty$.

$$\lim_{x\to0^+}\left(\frac{1}{2x}-\frac{1}{e^{2x}-1}\right)=\lim_{x\to0^+}\frac{e^{2x}-1-2x}{2x(e^{2x}-1)}\overset{\text{l'Ho}}{=}\lim_{x\to0^+}\frac{2e^{2x}-2}{2(e^{2x}-1)+4xe^{2x}}\overset{\text{l'Ho}}{=}\lim_{x\to0^+}\frac{4e^{2x}}{4e^{2x}+4e^{2x}+8xe^{2x}}=\frac{1}{2}.$$

j) 1^∞. Let $y=\displaystyle\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x$. We want to find y. Using the log process,

$$\begin{aligned}\ln y &= \ln\left[\lim_{x\to\infty}\left(1+\tfrac{k}{x}\right)^x\right]\\ \ln y &\overset{\text{Cont}}{=} \lim_{x\to\infty}\ln\left(1+\tfrac{k}{x}\right)^x\\ \ln y &= \lim_{x\to\infty}x\ln\left(1+\tfrac{k}{x}\right)\\ \ln y &= \lim_{x\to\infty}\frac{\ln\left(1+\frac{k}{x}\right)}{\frac{1}{x}}\\ \ln y &\overset{\text{l'Ho}}{=} \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\cdot\left(-\frac{k}{x^2}\right)}{-\frac{1}{x^2}}\\ \ln y &= \lim_{x\to\infty}\frac{k}{1+\frac{1}{x}}\\ \ln y &= k.\end{aligned}$$

But $\ln y=k$ implies $y=e^k$. So $\displaystyle\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x=y=k$.

k) 1^∞. Let $y=\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x^2}$. We want to find y. Using the log process,

$$\ln y = \ln\left[\lim_{x\to\infty}\left(1+\tfrac{1}{x}\right)^{x^2}\right]$$

$$\begin{aligned}
\ln y &\overset{\text{Cont}}{=} \lim_{x\to\infty} \ln\left(1+\tfrac{1}{x}\right)^{x^2}\\
\ln y &= \lim_{x\to\infty} x^2 \ln\left(1+\tfrac{1}{x}\right)\\
\ln y &= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x^2}}\\
\ln y &\overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)}{-\frac{2}{x^3}}\\
\ln y &= \lim_{x\to\infty} \frac{x}{1+\frac{1}{x}}\\
\ln y &= \infty.
\end{aligned}$$

But $\ln y = \infty$ implies "$y = e^{\infty}$" (Diverges). So $\lim\limits_{x\to\infty}\left(1+\dfrac{k}{x}\right)^x = \infty$ (Diverges).

l) $\frac{0}{0}$. $\lim\limits_{x\to 0} \dfrac{\arctan 4x}{\sin 2x} \overset{\text{l'Ho}}{=} \lim\limits_{x\to 0} \dfrac{\frac{4}{1+16x^2}}{2\cos 2x} = \dfrac{\frac{4}{1}}{2} = 2.$

m) Not indeterminate. $\lim\limits_{x\to 0} \dfrac{\sin 4x}{3\sec x} = \dfrac{0}{3} = 0.$ l'Hôpital's rule does not apply.

6. $\displaystyle\int_2^\infty \frac{4-x}{x^3+x^2-2x}\,dx = \lim_{b\to\infty}\int_2^b \frac{4-x}{x^3+x^2-2x}\,dx = \lim_{b\to\infty} -2\ln|x| + \ln|x-1| + \ln|x+2|\Big|_2^b$

$\displaystyle = \lim_{b\to\infty} \ln\left|\frac{x^2+x-2}{x^2}\right|\Bigg|_2^b = \lim_{b\to\infty}\ln\left|\frac{b^2+b-2}{b^2}\right| - \ln|1| = \lim_{b\to\infty}\ln\left|\frac{1+\frac{1}{b}-\frac{2}{b^2}}{1}\right| - \ln|1| = \ln|1| - \ln|1| = 0$

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7. a) Use substitution: $u = 4-x$ and $du = -dx$ to get $\displaystyle = -\int \frac{4-u}{u^{1/2}}\,du = -\int 4u^{-1/2} - u^{1/2}\,du = -8u^{1/2} + \frac{2}{3}u^{3/2} + c =$ $\displaystyle -8\sqrt{4-x} + \frac{2}{3}(4-x)^{3/2} + c.$

b) Use $u = 4-x^2$ to get $\int -\frac{1}{2}u^{-1/2}\,du = -u^{1/2} + c = -\sqrt{4-x^2} + c.$

c) Partial fractions: $\int \frac{4}{4-x^2}\,dx = \int \frac{1}{2+x} + \frac{1}{2-x}\,dx = \ln|2+x| - \ln|2-x| + c$

d) $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx = \int \frac{1}{\sqrt{2^2-x^2}}\,dx = \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{a} + c = \arcsin\frac{x}{2} + c.$

e) $\displaystyle\int \frac{4}{4+x^2}\,dx = 4\int \frac{1}{2^2+x^2}\,dx = 4\int \frac{1}{a^2+x^2}\,dx = \frac{4}{a}\arctan\frac{x}{a} + c = 2\arctan\frac{x}{2} + c.$

f) Use $u = 4-x^2$ to get $\int -\frac{2}{u}\,du = -2\ln|u| + c = -2\ln|4-x^2| + c.$

g) Triangle sub: $x = 2\sec\theta$, $dx = 2\sec\theta\tan\theta\,d\theta$, and $\sqrt{x^2-4} = 2\tan\theta$.

$$\ldots = \int \frac{1}{4\sec^2\theta 2\tan\theta} 2\sec\theta\tan\theta\,d\theta = \int \frac{1}{4}\cos\theta\,d\theta = \tfrac{1}{4}\sin\theta = \tfrac{\sqrt{4-x^2}}{4x} + c$$

h) Triangle sub: $x = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$, and $\sqrt{x^2-4} = 2\cos\theta$.

$$\ldots = \int \frac{4\sin^2\theta}{2\cos\theta}\cdot 2\cos\theta\,d\theta = \int 4\sin^2\theta\,d\theta = \int 2-2\cos 2\theta\,d\theta = 2\theta - \sin 2\theta = 2\theta - 2\sin\theta\cos\theta = 2\arcsin(\tfrac{x}{2}) - \tfrac{1}{2}x\sqrt{4-x^2} + c$$