Math 131 Lab 11

☞ Make sure you complete Problems 0–5(b) in Lab today.

0. Factorial Facts: Use the definition of factorial to simplify the following expressions. The last few will have n in the answer.

a) $\dfrac{9!}{7!}$ **b)** $\dfrac{8!}{5!3!}$ **c)** $\dfrac{(n-1)!}{(n+1)!}$ **d)** $\dfrac{(2n+2)!}{(2n)!}$

1. Review: Classify each integral as PROPER or IMPROPER. If it is improper, JUST REWRITE it using the correct limit expression (including one-sided limits, if necessary). Do NOT do any antidifferentiation. (How can you tell where the integrand is improper? Ask if you need help.)

a) $\displaystyle\int_0^1 \frac{x-1}{x^2-4x}\,dx$ **b)** $\displaystyle\int_2^5 \frac{x-1}{x^2-4x}\,dx$ **c)** $\displaystyle\int_2^3 \frac{x-1}{x^2-4x}\,dx$

2. a) Determine $\displaystyle\int \frac{2}{x^2-3x+2}\,dx$. Then evaluate each of the following definite integrals. Two of them are improper. Which two? Use correct limit notation on each of those. You may need to use log properties to help in the evaluation.

b) $\displaystyle\int_3^\infty \frac{2}{x^2-3x+2}\,dx$ **c)** $\displaystyle\int_3^5 \frac{2}{x^2-3x+2}\,dx$ **d)** $\displaystyle\int_2^3 \frac{2}{x^2-3x+2}\,dx$

3. Find the limits of the following sequences. Remember: If you use l'Hôpital's rule 'convert' the sequence to a corresponding function of x (because l'Hôpital's rule only applies to differentiable functions, not sequences). Use any appropriate **Key Limits** to avoid calculuation.

a) $\left\{\dfrac{3n}{n+1}\right\}_{n=1}^\infty$ **b)** $\left\{\dfrac{1-n^2}{6n+7}\right\}_{n=1}^\infty$ **c)** $\left\{\dfrac{e^n}{n^2+1}\right\}_{n=1}^\infty$ **d)** $\left\{(1+2n)^{\frac{1}{n}}\right\}_{n=1}^\infty$

e) $\left\{\dfrac{\ln\sqrt{n}}{n}\right\}_{n=1}^\infty$ **f)** $\left\{(\sqrt{n})^{\frac{1}{n}}\right\}_{n=1}^\infty$ **g)** $\left\{\left(1-\dfrac{1}{n}\right)^{\frac{n}{2}}\right\}_{n=1}^\infty$ **h)** $\{\ln(4n^2)-\ln(3n^2+n)\}_{n=1}^\infty$

i) $\left\{\dfrac{2n}{\sqrt{3n^2+1}}\right\}_{n=1}^\infty$ **j)** $\left\{\left(1-\dfrac{3}{n}\right)^{2n}\right\}_{n=1}^\infty$ **k)** $\left\{\dfrac{(n-1)!}{(n+1)!}\right\}_{n=1}^\infty$

l) $\{(-\pi)^n\}_{n=1}^\infty$ **m)** $\left\{\dfrac{(-2)^{4n}}{3^{3n}}\right\}_{n=1}^\infty$ **n)** $\{2^{3n}5^{-n}\}_{n=1}^\infty$ **o)** $\{\ln n^2+\ln\sin(2/n)\}_{n=1}^\infty$

4. Show that the following sequences are monotonic by using a derivative of an associated function.

a) $\{4^n\}_{n=1}^\infty$ **b)** $\left\{\dfrac{1-n}{n+2}\right\}_{n=1}^\infty$

5. Evaluate each integral. Caution: Which are improper?

a) $\displaystyle\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx$ **b)** $\displaystyle\int_2^4 \frac{1}{\sqrt{x^2-4}}\,dx$ **c)** $\displaystyle\int_{-\infty}^0 xe^{2x}\,dx$

6. Use #5(a) to find the average value of $\frac{1}{\sqrt{4-x^2}}$ on $[0,2)$. (Answer: $\pi/4$).

7. This is a beautiful problem which asks you to use many of the ideas of the course, including l'Hopital's rule and improper integrals. Consider the infinitely deep well formed by rotating the region bounded by the curves curve $y=\ln x$, the y-axis, and the x-axis in the fourth quadrant. it is filled with water (density $D = 62.5$ lbs/ft^3). How much work W is done in emptying the well (raising water to ground level $H = 0$.). The formula for work is

$$W = D\int_a^b (H-y)\pi(f(y))^2\,dy$$

where a and b the depths of the bottom and top of the well and $x=f(y)$ represents the equation of the curve as a function of x. Hint: Use your work in #5(c).

8. If you are still having trouble with improper integrals: Evaluate each improper integral; determine whether it converges or not. **Use proper notation! Suggestion:** Do the indefinite integral first and then use your work to evaluate the improper integral.

a) $\displaystyle\int_0^1 \frac{\ln x}{x}\,dx$ **b)** $\displaystyle\int_4^5 \frac{x}{\sqrt{x-4}}\,dx$ **c)** $\displaystyle\int_0^1 \frac{2}{x^2-1}\,dxr$

Some Answers

0. Use the definition of factorial and cancel common terms

a) 72 **b)** 56 **c)** $\dfrac{1}{n(n+1)}$ **d)** $(2n+1)(2n+2)$

1. Since $x^2 - 4x = 0$ when $x = 0$ or 4, the integral will be improper if it contains either of these points.

a) $\displaystyle\lim_{b\to 0^+}\int_b^1 \frac{x-1}{x^2-4x}\,dx$ **b)** $\displaystyle\lim_{b\to 4^-}\int_2^b \frac{x-1}{x^2-4x}\,dx + \lim_{b\to 4^+}\int_b^5 \frac{x-1}{x^2-4x}\,dx$ **c)** $\displaystyle\int_2^3 \frac{x-2}{x^2-4x}\,dx$ proper

2. a) $2\ln|x-1| + 2\ln|x-2| + c = 2\ln\left|\dfrac{x-2}{x-1}\right| + c$

b) $\ln 4$.

c) $\ln\frac{9}{16} - \ln\frac{1}{4} = \ln\frac{9}{4}$

d) $\lim_{b\to 2^+} 2\ln\left|\frac{1}{2}\right| - 2\ln\left|\frac{b-2^{\nearrow 0}}{b-1_{\searrow 1}}\right| = \infty$ (Diverges) Remember: $\lim_x \to 0 \ln|x| = -\infty$.

3. Make sure to convert to x when using L'Hopital.

a) 3 **b)** $-\infty$ Diverges **c)** ∞ Diverges **d)** 1

e) 0 **f)** 1 **g)** $\dfrac{1}{\sqrt{e}}$ **h)** $\ln(4/3)$

i) $\dfrac{2}{\sqrt{3}}$ **j)** e^{-6} **k)** 0

l) Diverges **m)** 0 **n)** Diverges **o)** Diverges

4. a) Increasing (non-decreasing). **(b)** Decreasing (non-increasing)

5. a) $\lim_{b\to 2^-}\arcsin\frac{b}{2} - \arcsin 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$ **(b)** Triangle: $\ln(2+\sqrt{3})$. **(c)** $-\frac{1}{4}$.

7. 15.625π ft – lbs.

8. a) $\lim_{a\to 0^+}\frac{1}{2}[(\ln 1)^2 - (\ln a)^2] = -\infty$. Diverges. **(b)** $8\frac{2}{3}$. **(c)** $\lim_{b\to 1^-}\ln\left|\frac{b-1}{b+1}\right| - \ln 1 = -\infty$. Diverges.

Summary of Key Limits

You should know and be able to use all of the following limits.

1. $\displaystyle\lim_{n\to\infty}\left(1+\tfrac{k}{n}\right)^n = e^k$. In particular $\displaystyle\lim_{n\to\infty}\left(1+\tfrac{1}{n}\right)^n = e$.

2. $\displaystyle\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty}\sqrt[n]{n} = 1$.

3. $\displaystyle\lim_{n\to\infty}\frac{n!}{n^n} = 0$ and $\displaystyle\lim_{n\to\infty}\frac{n^n}{n!} = \infty$ (diverges).

4. Consider the sequence $\{r^n\}_{n=1}^{\infty}$, where r is a real number.

a) If $|r| < 1$, then $\displaystyle\lim_{n\to\infty} r^n = 0$;

b) If $r = 1$, then $\displaystyle\lim_{n\to\infty} r^n = 1$;

c) Otherwise $\displaystyle\lim_{n\to\infty} r^n$ does not exist (diverges).

Math 131 Lab 11: Answers

0. Use the definition of factorial and cancel common terms

a) $\frac{9!}{7!} = \frac{1\cdot 2\cdots 7\cdot 8\cdot 9}{1\cdot 2\cdots 7} = 72$

b) $\frac{8!}{5!3!} = \frac{1\cdot 2\cdots 7\cdot 8}{1\cdot 2\cdots 5\cdot 1\cdot 2\cdot 3} = \frac{6\cdot 7\cdot 8}{6} = 56$

c) $\frac{(n-1)!}{(n+1)!} = \frac{1\cdot 2\cdots (n-1)}{1\cdot 2\cdots (n-1)\cdot n\cdot (n+1)} = \frac{1}{n(n+1)}$

d) $\frac{(2n+2)!}{(2n)!} = \frac{1\cdot 2\cdots (2n)\cdot (2n+1)\cdot (2n+2)}{1\cdot 2\cdots (2n)} = (2n+1)(2n+2)$

1. Since $x^2 - 4x = 0$ when $x = 0$ or 4, the integral will be improper if it contains either of these points.

a) $\lim_{b\to 0^+} \int_b^1 \frac{x-1}{x^2-4x}\,dx$ **b)** $\lim_{b\to 4^-} \int_2^b \frac{x-1}{x^2-4x}\,dx + \lim_{b\to 4^+} \int_b^5 \frac{x-1}{x^2-4x}\,dx$ **c)** $\int_2^3 \frac{x-2}{x^2-4x}\,dx$ proper

2. a) $\frac{1}{x^2-3x+2} = \frac{A}{x-1} + \frac{B}{x-2} = \frac{Ax - 2A + Bx - B}{(x-1)(x-2)}$.

$$\begin{array}{llll} x: & A+ & B & = 0 \\ \text{const:} & -2A & -B & = 2 \end{array} \Rightarrow A = -2,\ B = 2. \text{ So}$$

$$\int \frac{2}{x^2-3x+2}\,dx = \int \frac{-2}{x-1} + \frac{2}{x-2}\,dx = 2\ln|x-1| + 2\ln|x-2| + c = 2\ln\left|\frac{x-2}{x-1}\right| + c$$

b) $$\int_3^\infty \frac{2}{x^2-3x+2}\,dx = \lim_{b\to\infty}\int_3^b \frac{2}{x^2-3x+2}\,dx = \lim_{b\to\infty} 2\ln\left|\frac{x-2}{x-1}\right|\Bigg|_3^b = \lim_{b\to\infty} 2\ln\left|\frac{b-2}{b-1}\right| - 2\ln\left|\frac{1}{2}\right|$$

$$= \lim_{b\to\infty} 2\ln\left|\frac{1-\frac{2}{b}}{1-\frac{1}{b}}\right| - \ln\left|\frac{1}{4}\right| = \ln 1 - \ln\left|\frac{1}{4}\right| = \ln 4.$$

c) $\int_3^5 \frac{2}{x^2-3x+2}\,dx = 2\ln\left|\frac{x-2}{x-1}\right|\Bigg|_3^5 = 2\ln\left|\frac{3}{4}\right| - 2\ln\left|\frac{1}{2}\right| = \ln\frac{9}{16} - \ln\frac{1}{4} = \ln\frac{9}{4}$

d) $$\int_2^3 \frac{2}{x^2-3x+2}\,dx = \lim_{b\to 2^+}\int_b^3 \frac{2}{x^2-3x+2}\,dx = \lim_{b\to 2^+} 2\ln\left|\frac{x-2}{x-1}\right|\Bigg|_b^3 = \lim_{b\to 2^+} 2\ln\left|\frac{1}{2}\right| - 2\ln\left|\frac{b-2^{\nearrow 0}}{b-1_{\searrow 1}}\right| = \infty$$

(Diverges) Remember: $\lim_x \to 0 \ln|x| = -\infty$.

3. a) $\lim_{n\to\infty} \frac{3n}{n+1} = \lim_{n\to\infty} \frac{3}{1+\frac{1}{n}} = 3.$

b) $\lim_{n\to\infty} \frac{1-n^2}{6n+7} = \lim_{n\to\infty} \frac{\frac{1}{n}-n}{6+\frac{7}{n}} = -\infty$. Diverges.

c) $\lim_{n\to\infty} \frac{e^n}{n^2+1} = \lim_{x\to\infty} \frac{e^x}{x^2+1} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{e^x}{2x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{e^x}{2} = \infty$. Diverges.

d) ∞^0. Let $y = \lim_{n\to\infty} (1+2n)^{\frac{1}{n}}$, so

$$\begin{aligned} \ln y &= \ln\left[\lim_{n\to\infty}(1+2n)^{\frac{1}{n}}\right] = \ln\left[\lim_{x\to\infty}(1+2x)^{\frac{1}{x}}\right] = \lim_{x\to\infty} \ln\left[(1+2x)^{\frac{1}{x}}\right] = \lim_{x\to\infty} \frac{1}{x}\ln(1+2x) \\ \ln y &= \lim_{x\to\infty} \frac{\ln(1+2x)}{x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{\frac{2}{1+2x}}{1} = \frac{0}{1} = 0. \end{aligned}$$

But $\ln y = 0$ implies $y = 1$. So $y = \lim_{n\to\infty}(1+2n)^{\frac{1}{n}} = 1$.

e) $\lim_{n\to\infty} \frac{\ln\sqrt{n}}{n} = \lim_{x\to\infty} \frac{\ln\sqrt{x}}{x} = \lim_{x\to\infty} \frac{\frac{1}{2}\ln x}{x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{\frac{1}{2x}}{1} = 0$. Converges.

f) ∞^0. Let $\lim_{n\to\infty}(\sqrt{n})^{\frac{1}{n}} = \lim_{n\to\infty}\sqrt{n^{\frac{1}{n}}} \overset{\text{Key}}{=} \sqrt{1} = 1$

g) 1^∞. Key limit $y = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{\frac{n}{2}} = \lim_{n\to\infty}\left[\left(1-\frac{1}{n}\right)^{n}\right]^{1/2} \overset{\text{Key}}{=} 1^{1/2} = 1.$

h) $\infty - \infty$. $\lim_{n\to\infty}[\ln(4n^2) - \ln(3n^2+n)] = \lim_{n\to\infty} \ln\frac{4n^2}{3n^2+n} = \ln \lim_{n\to\infty}\frac{4}{3+\frac{1}{n}} = \ln\frac{4}{3}.$

i) ∞/∞. $\lim_{n\to\infty}\frac{2n}{\sqrt{3n^2+1}}$. Now divide by $\sqrt{n^2}$ on the bottom and $n=\sqrt{n^2}$ on the top: $= \lim_{n\to\infty}\frac{2}{\sqrt{3+\frac{1}{n^2}}} = \frac{2}{\sqrt{3}}.$

j) 1^∞. $\lim_{n\to\infty}\left(1-\frac{3}{n}\right)^{2n} = \lim_{n\to\infty}\left[\left(1-\frac{3}{n}\right)^{n}\right] = [e^{-3}]^2 = e^{-6}$. Key limit.

k) ∞/∞. $\lim_{n\to\infty}\frac{(n-1)!}{(n+1)!} = \lim_{n\to\infty}\frac{1}{n(n+1)} = 0$

l) $\lim_{n\to\infty}(-\pi)^n$ Diverges. Key limit (Geometric Sequence $|r|>1$)).

m) $\lim_{n\to\infty}\frac{(-2)^{4n}}{3^{3n}} = \lim_{n\to\infty}\frac{16^n}{27^n} = \lim_{n\to\infty}\left(\frac{16}{27}\right)^n = 0$ Diverges. Key limit (Geometric Sequence $|r|<1$).

n) $\lim_{n\to\infty} 2^{3n}5^{-n} = \lim_{n\to\infty}\frac{(2^3)^{n\ n}}{5} = \lim_{n\to\infty}\left(\frac{8}{5}\right)^n$ Diverges. Key limit (Geometric Sequence $|r|>1$)).

o) $\infty - \infty$. $\lim_{n\to\infty}\ln n^2 + \sin(2/n) = \lim_{n\to\infty}\ln n^2\sin(2/n) \overset{\text{Cont n to x}}{=} \ln\lim_{x\to\infty}\frac{\sin(2/x)}{\frac{1}{x^2}} \overset{\text{l'Ho}}{=} \ln\lim_{x\to\infty}\frac{-\frac{2}{x^2}\cos(2/x)}{-\frac{2}{x^3}} =$
$\lim_{n\to\infty}\ln x\cos(2/x)^{\nearrow \infty\cdot 1} = \infty$. Diverges

4. a) The corresponding function is $f(x)=4^x$. $f'(x)=\ln(4)4^x > 0$ for all x in the interval $[1,\infty)$. So $\{4^n\}_{n=1}^\infty$ is increasing (non-decreasing).

b) The corresponding function is $f(x)=\frac{1-x}{x+2}$. $f'(x)=\frac{-(x+2)-(1-x)}{(x+2)^2} = \frac{-3}{(x+2)^2} < 0$ for all x in the interval $[1,\infty)$. So $\left\{\frac{1-n}{n+2}\right\}_{n=1}^\infty$ is decreasing (non-increasing).

5. a) $\int\frac{1}{\sqrt{4-x^2}}\,dx = \int\frac{1}{\sqrt{2^2-x^2}}\,dx = \arcsin\frac{x}{2}$. So

$$\int_0^2\frac{1}{\sqrt{4-x^2}}\,dx = \lim_{b\to 2^-}\int_0^b\frac{1}{\sqrt{4-x^2}}\,dx = \lim_{b\to 2^-}\arcsin\frac{x}{2}\Big|_0^b = \lim_{b\to 2^-}\arcsin\frac{b}{2} - \arcsin 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

b) Triangle: $x = 2\sec\theta$. $\int\frac{1}{\sqrt{x^2-4}}\,dx = \int\frac{2\sec\tan\theta}{2\tan\theta}\,d\theta = \int\sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| = \ln|\frac{x}{2}+\frac{\sqrt{x^2-4}}{2}|$. So

$$\int_2^4\frac{1}{\sqrt{x^2-4}}\,dx = \lim_{a\to 2^+}\int_a^4\frac{1}{\sqrt{x^2-4}}\,dx = \lim_{a\to 2^+}\ln\left|\frac{x}{2}+\frac{\sqrt{x^2-4}}{2}\right|\Bigg|_a^4$$

$= \lim_{a\to 2^+}\ln|\frac{4}{2}+\frac{\sqrt{16-4}}{2}| - \ln|\frac{a}{2}+\frac{\sqrt{a^2-4}}{2}| = \ln|2+\sqrt{3}| - \ln|1| = \ln|2+\sqrt{3}|$

c) Parts: $u = x \Rightarrow du = dx$ and $dv = e^{2x}\,dx \Rightarrow v = \frac{1}{2}e^{2x}$ $\int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \int\frac{1}{2}e^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}$. So

$$\lim_{a\to-\infty}\int_a^0 xe^{2x}\,dx = \lim_{a\to-\infty}\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\Big|_a^0 = \lim_{a\to-\infty}\left(0-\frac{1}{4}\right) - \left(\frac{1}{2}ae^{2a} - \frac{1}{4}e^{2a}\right) = -\frac{1}{4}.$$

Use L'H for $\lim_{a\to-\infty} ae^{2a} = \lim_{a\to-\infty}\frac{a}{e^{-2a}} = \lim_{a\to-\infty}\frac{1}{-2e^{-2a}} = 0.$

6. $AV = \frac{1}{2-0}\int_0^2\frac{1}{\sqrt{4-x^2}}\,dx = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}.$

7. $y = \ln x$ so $x = e^y$. Use problem 3 (a).

$$W = 62.5\int_{-\infty}^0 (0-y)\pi(e^y)^2\,dy = \lim_{a\to-\infty} -62.5\pi\int_a^0 ye^{2y} = -62.5\pi/4 = 15.625\pi\ \text{ft}-\text{lbs}.$$

8. a) u-substitution: $u = \ln x$. Then $\int \frac{\ln x}{x}\,dx = \int u\,du = \frac{1}{2}u^2 + c = \frac{1}{2}(\ln x)^2 + c$. So

$$\int_0^1 \frac{\ln x}{x}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{\ln x}{x}\,dx = \lim_{a\to 0^+} \frac{1}{2}(\ln x)^2\Big|_a^1 = \lim_{a\to 0^+} \frac{1}{2}[(\ln 1)^2 - (\ln a)^2] = -\infty \text{ Diverges.}$$

b) u-sub: $u = x-4$. Then $\int \frac{x}{\sqrt{x-4}}\,dx = \int \frac{u+4}{u^{1/2}}\,du = \int u^{1/2} + 4u^{-1/2}\,du = \frac{2}{3}(x-4)^{3/2} + 8(x-4)^{1/2}$. So

$$\int_4^5 \frac{x}{\sqrt{x-4}}\,dx = \lim_{a\to 4^+} \int_a^5 \frac{x}{\sqrt{x-4}}\,dx = \lim_{a\to 4^+} \frac{2}{3}(x-4)^{3/2} + 8(x-4)^{1/2}\Big|_a^5 = \lim_{a\to 4^+} \frac{2}{3} + 8 - (0+0) = \frac{26}{3}.$$

c) Partial fractions: $\int \frac{2}{x^2-1}\,dx = \int \frac{1}{x-1} - \frac{1}{x+1}\,dx = \ln|x-1| - \ln|x+1| = \ln\left|\frac{x-1}{x+1}\right|$. So: $\displaystyle\int_0^1 \frac{2}{x^2-1}\,dx = \lim_{b\to 1^-} \ln\left|\frac{x-1}{x+1}\right|\Big|_0^b = \lim_{b\to 1^-} \ln\left|\frac{b-1}{b+1}\right| - \ln 1 = -\infty$. Diverges