Math 131 Day 35: Practest 3

Consult Labs 10, 11, and 12.

1. Find the average value of $f(x) = \frac{1}{x^2-9x+20}$ on the interval $[1, 3]$.

2. Find the volume of the infinitely long solid region generated when the area in the first quadrant enclosed by $y = \sqrt{\dfrac{8}{x^2+6x+5}}$, from $x = 0$ to $x = \infty$ is revolved around the x-axis. Use disks.

3. Find the volume of the solid region generated when the area enclosed by $y = \sqrt{\dfrac{8}{x^2+6x+5}}$, from $x = -1$ to $x = 0$ is revolved around the x-axis. Use disks.

4. a) Find the area in the first quadrant enclosed by $y = \frac{1}{\sqrt{x^2-9}}$ $y = 0$, and $x = 3$ and $x = 5$.

b) Find the area in the first quadrant enclosed by $y = \frac{1}{\sqrt{x^2-9}}$ $y = 0$, and $x = 5$ and $x = \infty$.

5. Try these; many are similar looking integrals. The first few are improper.

a) $\displaystyle\int_0^\infty \frac{4}{4+x^2}\,dx$ **b)** $\displaystyle\int_3^\infty \frac{4}{4-x^2}\,dx$ **c)** $\displaystyle\int_1^2 \frac{4}{4-x^2}\,dx$ **d)** $\displaystyle\int_3^\infty \frac{4x}{4-x^2}\,dx$

e) $\displaystyle\int \frac{4x+1}{x^2-5x+4}\,dx$ **f)** $\displaystyle\int \frac{4x+8}{x^2+4x+5}\,dx$

g) $\displaystyle\int \frac{-4x+4}{(x-2)^2x}\,dx$ **h)** $\displaystyle\int \frac{4x+1}{x^2-4}\,dx$ **i)** $\displaystyle\int \frac{4}{(4-x)^{2/3}}\,dx$

j) $\displaystyle\int \frac{8x+4}{x^3+x^2-2x}\,dx$ **k)** $\displaystyle\int_0^2 \frac{2x+6}{x^2+2x-8}\,dx$

l) $\displaystyle\int_4^\infty \frac{-3}{x^2-3x}$ **m)** $\displaystyle\int \frac{4x^2+8x+2}{x(x+1)^2}\,dx$ **n)** $\displaystyle\int \frac{4x}{(x+1)^3}\,dx$

6. Use L'Hopital's Rule if appropriate. (Answers **not** in order: 0, 0, 0, $\frac{1}{2}$, 1, 1 ln 2, 2, e, 3, 4, 5, -6, e^7.)

a) $\displaystyle\lim_{x\to1} \frac{x-1}{\ln x}$ **b)** $\displaystyle\lim_{x\to0} \frac{x^2+4x}{\sin 2x}$ **c)** $\displaystyle\lim_{x\to\infty} \frac{6x^2+3x-1}{2x^2+x}$

d) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{e^x}$ **e)** $\displaystyle\lim_{x\to1} \frac{6x+10}{3x+1}$ **f)** $\displaystyle\lim_{x\to0} \frac{\tan 5x}{\arcsin x}$ **g)** $\displaystyle\lim_{x\to\infty} \left(1+\tfrac{7}{x}\right)^x$

h) $\displaystyle\lim_{x\to0} \frac{\cos 4x - \cos 2x}{x^2}$ **i)** $\displaystyle\lim_{x\to0} \frac{e^x-1-x}{x^2}$ **j)** $\displaystyle\lim_{x\to0^+} 2x\ln x$ **k)** $\displaystyle\lim_{x\to\infty} x^2e^{-x}$

l) $\displaystyle\lim_{x\to\infty} \ln(2x+9)-\ln(x+7)$ **m)** $\displaystyle\lim_{x\to0^+} (2x)^x$ **n)** $\displaystyle\lim_{x\to0^+} (1+x)^{1/x}$

7. Find the limits of these sequences. Use the key limits when possible (indicate when you do so). For the last part, use the derivative formula: $\frac{d}{dx}(a^x) = a^x \ln a$, when $a > 0$.

a) $\displaystyle\left\{\left(1+\frac{3}{n}\right)^n\right\}_{n=1}^\infty$ **b)** $\displaystyle\{\ln(2n^2+7)-\ln(5n^2+n)\}_{n=1}^\infty$ **c)** $\displaystyle\left\{\frac{2\ln(n+1)}{n^2}\right\}_{n=1}^\infty$

d) $\displaystyle\left\{\left(\frac{2}{3}\right)^n\right\}_{n=1}^\infty$ **e)** $\displaystyle\left\{\left(\frac{-3}{2}\right)^n\right\}_{n=1}^\infty$ **f)** $\displaystyle\left\{\frac{4n^2-3n+1}{5n^2+7}\right\}_{n=1}^\infty$

g) $\displaystyle\left\{\left(\frac{n^2}{3^n}\right)\right\}_{n=1}^\infty$

8. Find the sums of these series, if they exist. Note the starting indices! Two are telescoping.

a) $\displaystyle\sum_{n=0}^\infty 4\left(\frac{-2}{5}\right)^n$ **b)** $\displaystyle\sum_{n=0}^\infty 4\left(\frac{10}{9}\right)^n$ **c)** $\displaystyle\sum_{n=1}^\infty \frac{2}{n^2+n}$

d) $\displaystyle\sum_{n=0}^\infty \frac{6}{n^2+7n+12}$ **e)** $\displaystyle\sum_{n=3}^\infty 5\left(\frac{1}{2}\right)^n$ **f)** $\displaystyle\sum_{n=1}^\infty 3\left(\frac{-2}{3}\right)^{3n}$

9. Determine whether the following series converge. First determine which test to use: nth term test, p-series test, integral test, geometric series test, or the comparison tests. Your final answer should consist of a little 'argument' (a sentence or two) and any necessary calculations. **Use appropriate mathematical language**.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^7}}$ **b)** $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{1+e^n}$ **c)** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{16+9n^2}$ **d)** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{n^2}$ **e)** $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n^{1/3}}$

f) $\displaystyle\sum_{n=1}^{\infty} \ln(3n+3)-\ln(6n+2)$ **g)** $\displaystyle\sum_{n=1}^{\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}}$ **h)** $\displaystyle\sum_{n=1}^{\infty} 2\left(\frac{-3}{11}\right)^n$ **i)** $\displaystyle\sum_{n=2}^{\infty} \frac{n}{\ln n}$ **j)** $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^n}$

k) $\displaystyle\sum_{n=1}^{\infty} 6\left(\frac{5}{4}\right)^n$ **l)** $\displaystyle\sum_{n=0}^{\infty} \frac{3n^2}{n^3+1}$ **m)** $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+1)!}{(2n-1)!}$ **n)** $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^2+7n+10}$ **o)** $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

10. Now try these

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$ **b)** $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{4^n}$ **c)** $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{n!}$ **d)** $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ **e)** $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1}}{n^n}$

Brief Answers

More complete answers to follow.

1. $\frac{1}{2}(\ln 3+\ln 2-\ln 4)$.

2. Ans: $2\pi\ln 5$.

3. Improper (at -1). Ans: Diverges.

4. a) $\ln 3$
b) Ans: Diverges

5. Watch for typos. If you find any, let me know. The indefinite integrals are all $+c$.

a) π **b)** $-\ln 5$ **c)** diverges **d)** diverges

e) $\frac{17}{3}\ln|x-4|-\frac{5}{3}\ln|x-1|$ **f)** $2\ln|x^2+4x+5|$

g) $\dfrac{2}{x-2}+\ln\left|\dfrac{x}{x-2}\right|$ **h)** $2\sqrt{3}-2\pi/3$

i) $\frac{9}{4}\ln|x-2|+\frac{7}{4}\ln|x+2|$ **j)** $4\ln|x-1|-2\ln|x|-2\ln|x+2|$ **k)** $-\infty$ Diverges

l) $\lim_{b\to\infty}\ln\left|\frac{x}{x-3}\right|\Big|_4^b=-\ln|4|$ **m)** $2\ln|x|-\dfrac{2}{x+1}+2\ln|x+1|$ **n)** $-4(x+1)^{-1}+2(x+1)^{-2}$

Math 131 Practest 3: Answer Sketches

1. $AV = \frac{1}{3-1}\int_1^3 \frac{1}{x^2-9x+20}\,dx = \int_1^3 \frac{1/2}{x-5} - \frac{1/2}{x-4}\,dx = \frac{1}{2}(\ln|x-5| - \ln|x-4|)\big|_1^3 = \frac{1}{2}(\ln 2 - \ln 4 + \ln 3)$.

2. $\frac{8}{x^2+6x+5} = \frac{2}{x+1} - \frac{2}{x+5}$. So $\int_0^\infty \frac{8}{x^2+6x+5}\,dx = \int \frac{2}{x+1} - \frac{2}{x+5}\,dx = 2\pi\ln|\frac{x+1}{x+5}|$.

$$V = \pi\int_0^\infty \frac{8}{x^2+6x+5}\,dx = \lim_{b\to\infty} 2\pi\ln|\tfrac{x+1}{x+5}|\big|_0^b = \lim_{b\to\infty} 2\pi[\ln|\tfrac{b+1}{b+5}| - \ln|\tfrac{1}{5}|] = 2\pi[\ln|1| - \ln|\tfrac{1}{5}|] = 2\pi\ln 5.$$

3. As above:

$$V = \pi\int_{-1}^0 \frac{8}{x^2+6x+5}\,dx = \lim_{a\to -1^+} 2\pi\ln\left|\frac{x+1}{x+5}\right|\Bigg|_a^0 = \lim_{a\to -1^+} 2\pi\left[\ln\left|\frac{1}{5}\right| - \ln\left|\frac{a+1}{a+5}\right|\right] = 2\pi\left[\ln\left|\frac{1}{5}\right| - \infty\right]. \text{ Diverges.}$$

4. a) Triangles. $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\,d\theta$, $\sqrt{x^2-9} = 3\tan\theta$.

$$\int \frac{1}{\sqrt{x^2-9}}\,dx = \int \frac{3\sec\theta\tan\theta}{3\tan\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right|$$

$$\int_3^5 \frac{1}{\sqrt{x^2-9}}\,dx = \lim_{a\to 3^+} \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right|\Bigg|_a^5 = \lim_{a\to 3^+} \ln\left|\frac{5}{3} + \frac{4}{3}\right| - \ln\left|\frac{a}{3} + \frac{\sqrt{a^2-9}}{3}\right| = \ln|3| - \ln|1| = \ln 3.$$

b) Use the work above.

$$\int_5^\infty \frac{1}{\sqrt{x^2-9}}\,dx = \lim_{b\to\infty} \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right|\Bigg|_5^b = \lim_{b\to\infty} \ln\left|\frac{b}{3} + \frac{\sqrt{b^2-9}}{3}\right| - \ln\left|\frac{5}{3} + \frac{4}{3}\right| = \infty - \ln|3|. \text{ Diverges}$$

5. a) $\lim_{b\to\infty} 4\int_0^b \frac{1}{2^2+x^2}\,dx = \lim_{b\to\infty} 4\cdot\frac{1}{2}\arctan\frac{x}{2}\big|_0^b = \lim_{b\to\infty} 2\arctan\frac{b}{2} - 0 = 2(\frac{\pi}{2}) - 0 = \pi$.

b) $\lim_{b\to\infty}\int_3^\infty \frac{4}{4-x^2}\,dx = \lim_{b\to\infty}\int_3^b \frac{1}{2-x} + \frac{1}{2+x}\,dx = \lim_{b\to\infty}(-\ln|2-x| + \ln|2+x|)\big|_3^b = \lim_{b\to\infty}\ln|\frac{2+x}{2-x}|\big|_3^b = \lim_{b\to\infty}\ln|\frac{2+b}{2-b}| - \ln 5 = \ln 1 - \ln 5 = -\ln 5$.

c) $\lim_{b\to 2^-}\int_1^b \frac{4}{4-x^2}\,dx = \lim_{b\to 2^-}\ln|\frac{2+x}{2-x}|\big|_1^b = \lim_{b\to 2^-}\ln|\frac{2+b}{2-b}| - \ln 3 = +\infty$. Diverges.

d) u-sub: $\lim_{b\to\infty}\int_3^b \frac{4x}{4-x^2}\,dx = \lim_{b\to\infty}(-2\ln|4-x^2|)\big|_3^b = \lim_{b\to\infty}\ln|4-b^2| - \ln 5 = +\infty$. Diverges.

e) $\int \frac{4x+1}{x^2-5x+4}\,dx = \int \frac{17/3}{x-4} - \frac{5/3}{x-1}\,dx = \frac{17}{3}\ln|x-4| - \frac{5}{3}\ln|x-1| + c$.

f) Not partial fractions. Can't factor. u-substitution: $u = x^2+4x+5$, $du = 2x+4\,dx$. $\int \frac{4x+8}{x^2+4x+5}\,dx = \int \frac{2}{u}\,du = 2\ln|u| + c = 2\ln|x^2+4x+5| + c$.

g) $\int -\frac{1}{x-2} - \frac{2}{(x-2)^2} + \frac{1}{x}\,dx = -\ln|x-2| + 2(x-2)^{-1} + \ln|x| + c$.

h) $\int \frac{4x+1}{x^2-4}\,dx = \int \frac{9/4}{x-2} + \frac{7/4}{x+2}\,dx = \frac{9}{4}\ln|x-2| + \frac{7}{4}\ln|x+2| + c$.

i) $u = 4-x$, $du = -dx$. $\int -4u^{-2/3}\,du = -12u^{1/3} + c = -12(4-x)^{-1/3} + c$.

j) $\int \frac{4}{x-1} - \frac{2}{x} - \frac{2}{x+2}\,dx = 4\ln|x-1| - 2\ln|x| - 2\ln|x+2| + c$.

k) $\int \frac{2x+6}{x^2+2x-8}\,dx = \int \frac{5/3}{x-2} + \frac{1/3}{x+4}\,dx = \frac{5}{3}\ln|x-2| + \frac{1}{3}\ln|x+4|$. So

$$\int_0^2 \frac{2x+6}{x^2+2x-8}\,dx = \lim_{b\to 2^-}\frac{5}{3}\ln|x-2| + \frac{1}{3}\ln|x+4|\Bigg|_0^b = \lim_{b\to 2^-}\frac{5}{3}\ln|b-2| + \frac{1}{3}\ln|b+4| - \left[\frac{5}{3}\ln|-2| + \frac{1}{3}\ln|4|\right]$$

$$= -\infty + \frac{1}{3}\ln|6| - \left[\frac{5}{3}\ln|-2| + \frac{1}{3}\ln|4|\right]. \text{ Diverges}$$

l) $\int \frac{-3}{x^2-3x}\,dx = \int \frac{1}{x} - \frac{1}{x-3}\,dx = \ln|x| - \ln|x-3| = \ln|\frac{x}{x-3}|$. So

$$\int_4^\infty \frac{3}{x^2-3x}\,dx = \lim_{b\to\infty}\ln\left|\frac{x}{x-3}\right|\Bigg|_4^b = \lim_{b\to\infty}\ln\left|\frac{b}{b-3}\right| - \ln|4| = \ln|1| - \ln|4| = -\ln|4|.$$

m) $\int \frac{4x^2+8x+2}{x(x+1)^2}\,dx = \int \frac{2}{x} + \frac{2}{x+1} + \frac{2}{(x+1)^2}\,dx = 2\ln|x| + 2\ln|x+1| - 2(x+1)^{-1} + c.$

n) $\dfrac{4x}{(x+1)^3} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x+1)^3} = \dfrac{Ax^2 + 2Ax + A + Bx - B + C}{(x+1)^3}.$

$$\begin{array}{llll} x^2: & A & & = 0 \\ x: & 2A+ & B & = 4 \\ \text{const:} & & B+ \quad C & = 0 \end{array} \Rightarrow A = 0,\ B = 4,\ C = -4.\ \int \frac{4}{(x+1)^2} - \frac{4}{(x+1)^3}\,dx = -4(x+1)^{-1} + 2(x+1)^{-2} + c.$$

6. a) $\lim_{x\to 1} \frac{x-1}{\ln x} \overset{\text{l'Ho}}{=} \lim_{x\to 1} \frac{1}{1/x} = 1$

b) $\lim_{x\to 0} \frac{x^2+4x}{\sin 2x} \overset{\text{l'Ho}}{=} \lim_{x\to 0} \frac{2x+4}{2\cos 2x} = \frac{4}{2} = 2$

c) $\lim_{x\to\infty} \frac{6x^2+3x-1}{2x^2+x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{12x+3}{4x+1} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{12}{4} = 3$

d) $\lim_{x\to\infty} \frac{\ln x}{e^x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{1/x}{e^x} = 0$

e) $\lim_{x\to 1} \frac{6x+10}{3x+1} = \frac{16}{4} = 4$

f) $\lim_{x\to 0} \frac{\tan 5x}{\arcsin x} \overset{\text{l'Ho}}{=} \lim_{x\to 0} \frac{5\sec^2 5x}{\frac{1}{\sqrt{1-x^2}}} = 5$

g) Recognize as a key limit: e^7. OR: assume $y = \lim_{x\to\infty}\left(1+\frac{7}{x}\right)^x$.

$$\ln y = \ln \lim_{x\to\infty}\left(1+\frac{7}{x}\right)^x = \lim_{x\to\infty} x\ln(1+\frac{7}{x}) = \lim_{x\to\infty}\frac{\ln(1+\frac{7}{x})}{\frac{1}{x}} \overset{\text{l'Ho}}{=} \lim_{x\to\infty}\frac{\frac{1}{1+\frac{7}{x}}\cdot\frac{-7}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{1}{1+\frac{7}{x}}\cdot 7 = 7.$$

So $y = e^7 = \lim_{x\to\infty}(1+\frac{7}{x})^x$.

h) $\lim_{x\to 0} \frac{\cos 4x - \cos 2x}{x^2} \overset{\text{l'Ho}}{=} \lim_{x\to 0} \frac{-4\sin 4x + 2\sin 2x}{2x} \overset{\text{l'Ho}}{=} \lim_{x\to 0} \frac{-16\cos 4x + 4\cos 2x}{2} = -\frac{12}{2} = -6.$

i) $\lim_{x\to 0} \frac{e^x-1-x}{x^2} \overset{\text{l'Ho}}{=} \lim_{x\to 0} \frac{e^x-1}{2x} \overset{\text{l'Ho}}{=} \lim_{x\to 0} \frac{e^x}{2} = 1/2$

j) $\lim_{x\to 0^+} 2x\ln x \overset{\text{l'Ho}}{=} \lim_{x\to 0^+} \frac{2\ln x}{1/x} \overset{\text{l'Ho}}{=} \lim_{x\to 0^+} \frac{2/x}{-1/x^2} = \lim_{x\to 0^+} -\frac{2x^2}{x} = \lim_{x\to 0^+} -2x = 0$

k) $\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$

l) Use logs: $= \lim_{x\to\infty} \ln\frac{2x+9}{x+7}$. Use continuity: $= \ln(\lim_{x\to\infty} \frac{2x+9}{x+7}) \overset{\text{l'Ho}}{=} \ln(\frac{2}{1}) = \ln 2$

m) Assume $y = \lim_{x\to 0^+}(2x)^x$. Then $\ln y = \lim_{x\to 0^+} \ln(2x)^x = \lim_{x\to 0^+} \frac{\ln(2x)}{1/x} \overset{\text{l'Ho}}{=} \lim_{x\to 0^+} \frac{2/2x}{-1/x^2} = \lim_{x\to 0^+} -\frac{x^2}{x} = \lim_{x\to 0^+} -x = 0$. So $\ln y = 0$, therefore $y = e^0 = 1 = \lim_{x\to 0^+}(2x)^x$.

n) Assume $y = \lim_{x\to 0^+}(1+x)^{1/x}$. As above $\ln y = \lim_{x\to 0^+} \ln(1+x)^{1/x} = \lim_{x\to 0^+} \frac{\ln(1+x)}{x} \overset{\text{l'Ho}}{=} \lim_{x\to 0^+} \frac{1/(1+x)}{1} = 1$. Therefore $y = e^1 = e = \lim_{x\to 0^+}(1+x)^{1/x}$.

7. Find the limits of these sequences. Use the key limits when possible. When using l'Hopital's rule, switch to x.

a) $\displaystyle\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^n = e^3$ (key limit)

b) $\displaystyle\lim_{n\to\infty} \ln(2n^2+7) - \ln(5n^2+n) = \lim_{x\to\infty} \ln\frac{2x^2+7}{5x^2+n} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \ln\frac{4x}{10x+1} \overset{\text{l'Ho}}{=} \ln\frac{4}{10}$

c) $\displaystyle\lim_{n\to\infty} \frac{2\ln(n+1)}{n^2} = \lim_{x\to\infty}\frac{2\ln(x+1)}{x^2} \overset{\text{l'Ho}}{=} \lim_{x\to\infty}\frac{\frac{2}{x+1}}{2x} = 0$

d) $\displaystyle\lim_{n\to\infty}\left(\frac{2}{3}\right)^n = 0$ (key limit)

e) $\displaystyle\lim_{n\to\infty}\left(\frac{-3}{2}\right)^n$ diverges (key limit)

f) $\displaystyle\lim_{n\to\infty}\frac{4n^2-3n+1}{5n^2+7} = \lim_{n\to\infty}\frac{4-\frac{3}{n}+\frac{1}{n^2}}{5+\frac{7}{n^2}} = \frac{4}{5}$

g) $\displaystyle\lim_{n\to\infty}\frac{n^2}{3^n} = \lim_{x\to\infty}\frac{x^2}{3^x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty}\frac{2x}{3^x\ln 3} \overset{\text{l'Ho}}{=} \lim_{x\to\infty}\frac{2}{3^x(\ln 3)^2} = 0.$

8. a) $\displaystyle\sum_{n=0}^{\infty} 4\left(\frac{-2}{5}\right)^n = \frac{4}{1-(-\frac{2}{5})} = \frac{20}{7}$

b) $\displaystyle\sum_{n=0}^{\infty} 4\left(\frac{10}{9}\right)^n$ diverges since $|r| = \dfrac{10}{9} > 1$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+n} = \sum_{n=1}^{\infty} \frac{2}{n} - \frac{2}{n+1}$. $S_n = (2-\frac{2}{2}) + (\frac{2}{2}-\frac{2}{3}) + \cdots + (\frac{2}{n} - \frac{2}{n+1}) = 2 - \frac{2}{n+1}$.

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 2 - \frac{2}{n+1} = 2$$

d) $\displaystyle\sum_{n=0}^{\infty} \frac{6}{n^2+7n+12} = \sum_{n=0}^{\infty} \frac{6}{n+3} - \frac{6}{n+4}$. $S_n = (\frac{6}{3}-\frac{6}{4}) + (\frac{6}{4}-\frac{6}{6}) + \cdots + (\frac{6}{n+3} - \frac{6}{n+4}) = 2 - \frac{6}{n+6}$.

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 2 - \frac{6}{n+4} = 2$$

e) $\displaystyle\sum_{n=3}^{\infty} 5\left(\frac{1}{2}\right)^n = 5\left(\frac{1}{2}\right)^3 + 5\left(\frac{1}{2}\right)^4 + 5\left(\frac{1}{2}\right)^5 + \cdots$. So a $= \dfrac{5}{8}$, r $= \dfrac{1}{2}$. So $\dfrac{\text{a}}{1-\text{r}} = \dfrac{\frac{5}{8}}{1-\frac{1}{2}} = \dfrac{5}{4}$.

f) $\displaystyle\sum_{n=1}^{\infty} 3\left(\frac{-2}{3}\right)^{3n} = \sum_{n=1}^{\infty} 3\left(\frac{-8}{27}\right)^n = 3\left(\frac{-8}{27}\right)^1 + 3\left(\frac{-8}{27}\right)^2 + \cdots$. So a $= \dfrac{-8}{9}$, r $= \dfrac{-8}{27}$. So $\dfrac{\text{a}}{1-\text{r}} = \dfrac{\frac{-8}{9}}{1+\frac{8}{27}} = -\dfrac{24}{35}$.

9. a) p-series with $p = 7/3 > 1$. The series series converges.

b) nth term test: $\displaystyle\lim_{n\to\infty} \frac{e^n}{1+e^n} = \lim_{x\to\infty} \frac{e^x}{1+e^x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{e^x}{e^x} = 1 \neq 0$. By the nth term test the series diverges. [Also integral test.]

c) Use the integral test. The corresponding function is $f(x) = \frac{2x}{1+x^4}$ and is clearly positive and continuous on $[1, \infty)$. Notice that $f'(x) = \frac{2-6x^4}{(1+x^4)^2} < 0$ for all x in $[1, \infty)$. So f is decreasing. The improper integral is $\displaystyle\int_1^{\infty} \frac{2x}{1+x^4}\,dx$. (recognize as an inverse tangent integral). $u^2 = x^4$, $u = x^2$, $du = 2x\,dx$ So $\frac{2x}{1+x^4}\,dx = \int \frac{1}{1+u^2}\,du = \arctan(u) + c = \arctan(x^2) + c$.

$$\int_1^{\infty} \frac{2x}{1+x^4}\,dx = \lim_{b\to\infty} \arctan(x^2)\Big|_1^b = \lim_{b\to\infty} \arctan(b^2) - \arctan(1) = [\tfrac{\pi}{2} - \tfrac{\pi}{4}] = \tfrac{\pi}{4}.$$

Since the integral converges, by the integral test the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{1+n^4}$ also converges.

d) nth term test: $\displaystyle\lim_{n\to\infty} \frac{2^n+1}{n^2} = \lim_{x\to\infty} \frac{2^x+1}{x^2} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{\ln 2 \cdot 2^x}{2x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{(\ln 2)^2 2^x}{2} = \infty \neq 0$. By the nth term test the series diverges.

e) nth term test: $\displaystyle\lim_{n\to\infty} \ln(3n+3) - \ln(6n+2) = \lim_{n\to\infty} \ln\frac{(3n+3)}{(6n+2)} = \ln 12 \neq 0$. By the nth term test the series diverges.

f) p-series with $p = 1/2 < 1$. The series series diverges.

g) nth term test: $\displaystyle\lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{-\frac{1}{x}\cdot\cos\frac{1}{x}}{-\frac{1}{x^2}} = \cos 0 = 1 \neq 0$. By the nth term test the series diverges.

h) Geometric series with $|r| = 3/11 < 1$. The series converges.

i) nth term test: $\displaystyle\lim_{n\to\infty} \frac{n}{\ln n} = \lim_{x\to\infty} \frac{x}{\ln x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{1}{1/x} = \infty \neq 0$. By the nth term test the series diverges.

j) Ratio Test (exponentials). The terms are positive and

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{3^n} = \lim_{n\to\infty} \frac{3n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{3}{n+1} \cdot \frac{n^n}{(n+1)^n}$$

$$= \lim_{n\to\infty} \frac{3}{n+1}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{3}{n+1}\left(\frac{1}{1+\frac{1}{n}}\right)^n = \lim_{n\to\infty} 0 \cdot \frac{1}{e} = 0.$$

By the ratio test the series converges.

k) Geometric series with $|r| = 5/4 \geq 1$. The series diverges.

l) Integral test: Note that $f(x) = \frac{3x^2}{x^3+1}$ is certainly positive and continuous; it is also decreasing: $f'(x) = \frac{1-3x^4}{(x^3+1)^2} < 0$ on $[1, \infty)$. Aside $u = x^3 + 1$, $du = 3x^2\,dx$. Then $\int \frac{3x^2}{x^3+1}\,dx = \int \frac{1}{u}\,du = \ln|u| = \ln|x^3+1|$. So

$$\lim_{b\to\infty} \int_0^b \frac{3x^2}{x^3+1}\,dx = \lim_{b\to\infty} \ln|x^3+1|\Big|_0^b = \lim_{b\to\infty} \ln|b^3+1| - \ln 1 = \infty$$

So $\sum \frac{3n^2}{n^3+1}$ also diverges by the integral test.

m) nth term test: $\lim\limits_{n\to\infty} \frac{(2n+1)!}{(2n-1)!} = \lim\limits_{n\to\infty} (2n+1)(2n) = \infty \neq 0$. By the nth term test the series diverges.

n) The integral test. Let $f(x) = \frac{3}{x^2+7x+10}$. Then $f(x)$ is positive and continuous on $[1, \infty)$. $f(x)$ is also decreasing because as x increases, the denominator increases and the numerator stays the same, so the overall function decreases. Use partial fractions (check that these are the correct fractions).

$$\int_2^\infty \frac{3}{x^2+7x+10}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x+2} - \frac{1}{x+5}\,dx = \lim_{b\to\infty} \ln|x+2| - \ln|x+5|\Big|_1^b = \lim_{b\to\infty} \ln\frac{x+2}{x+5}\Big|_1^b$$

$$= \lim_{b\to\infty} \ln\frac{b+2}{b+5} - \ln\frac{3}{6} = \lim_{b\to\infty} \ln\frac{1+2/b}{1+5/b} - \ln\frac{1}{2} = \ln 1 - \ln\frac{1}{2} = \ln 2.$$

Since the integral converges, so does the corresponding series $\sum_{n=1}^\infty \frac{3}{n^2+7n+10}$ by the integral test.

o) nth term test: $\lim\limits_{n\to\infty} \frac{n^n}{n!} \overset{\text{KeyLimit}}{=} \infty \neq 0$. By the nth term test the series diverges. OK by the Ratio Test, too: Positive terms and $r = \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{(n+1)^n}{(n+1)!} \cdot \frac{n!}{n^n} = \lim\limits_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)n^n} = \lim\limits_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim\limits_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e > 1$. So the series diverges by the ratio test.

10. a) Ratio Test (factorials). The terms are positive and $r = \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{1}{(n+1)!} \cdot \frac{n!}{1} = \lim\limits_{n\to\infty} \frac{1}{n+1} = 0 < 1$. By the ratio test the series converges.

b) Ratio Test (exponential). The terms are positive and $r = \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{(n+1)^4}{4^{n+1}} \cdot \frac{4^n}{n^4} = \lim\limits_{n\to\infty} \left(\frac{n+1}{n}\right)^4 \cdot \frac{1}{4} = \frac{1}{4} < 1$. By the ratio test the series converges. Better: Root Test (powers). The terms are positive and $r = \lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} \sqrt[n]{\frac{n^4}{4^n}} = \lim\limits_{n\to\infty} \frac{(\sqrt[n]{n})^4}{\sqrt[n]{4^n}} = \frac{1^4}{4} < 1$. By the root test the series converges.

c) Ratio Test (factorials). The terms are positive and $r = \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{\frac{(n+1)^4}{(n+1)!}}{\frac{n^4}{n!}} = \lim\limits_{n\to\infty} \frac{n!}{(n+1)!} \cdot \left(\frac{n+1}{n}\right)^4 = \lim\limits_{n\to\infty} \frac{1}{n+1} \cdot \left(\frac{1+1/n}{1}\right)^4 = 0 < 1$. By the ratio test the series converges.

d) Ratio Test (factorials). The terms are positive and $r = \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{\frac{([n+1]!)^2}{(2n+2)!}}{\frac{(n!)^2}{(2n)!}} = \lim\limits_{n\to\infty} \frac{([n+1]!)^2}{(n!)^2} \cdot \frac{(2n)!}{(2n+2)!} = \lim\limits_{n\to\infty} \frac{(n+1)^2}{(2n+1)(2n+2)} = \lim\limits_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2} = \frac{1}{4} < 1$. By the ratio test the series converges.

e) Ratio Test. The terms are positive and $r = \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{2^{n+2}}{(n+1)^{n+1}} \cdot \frac{n^n}{2^{n+1}} = \lim\limits_{n\to\infty} \frac{2n^n}{(n+1)^{n+1}} = \lim\limits_{n\to\infty} 2 \cdot \frac{n^n}{(n+1)^n} \cdot \frac{1}{n+1} = \lim\limits_{n\to\infty} \left(\frac{n}{n+1}\right)^n \cdot \frac{2}{n+1} = \lim\limits_{n\to\infty} \left(\frac{1}{1+1/n}\right)^n \cdot \frac{2}{n+1} = \lim\limits_{n\to\infty} \frac{1}{e} \cdot 0 = 0 < 1$. By the ratio test the series converges.