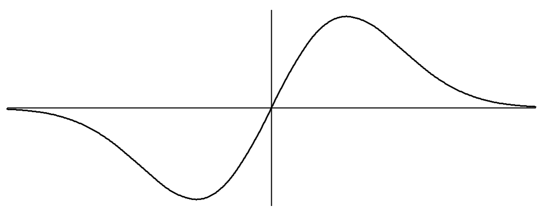
Math 131 Review Session

Warning: There may be material on the test not covered here and there may be material here that is not covered on the exam. See the Day 42 Handout for Details.

1. Let f be the function whose graph is given below. Use the information in the table, properties of the integral, and the **shape** of f to evaluate the given integrals.

a) $\int_3^0 f(x)\,dx$ **b)** $\int_{-1}^2 f(x)\,dx$ **c)** $\int_1^2 x + f(x)\,dx$ **d)** $\int_1^2 f(x-1)\,dx$



$\int_0^1 f(x)\,dx = 0.4$
$\int_0^2 f(x)\,dx = 0.8$
$\int_0^3 f(x)\,dx = 0.9$
$\int_0^4 f(x)\,dx = 1.0$

2. Find the arc length of $\ln(\cos x)$ on the interval $[0, \pi/3]$. Ans: $\ln|2+\sqrt{3}|$

3. Determine whether the alternating series $\sum_{n=2}^{\infty}(-1)^n\left(\frac{\sqrt{n}}{n-1}\right)$ converges. Carefully check whether $a_{n+1} \le a_n$.

4. Integral Mix Up: First classify each by the technique that you think will apply.

a) $\int \sec^2(x)e^{\tan x}\,dx$ **b)** $\int x\sec^2 x\,dx$ **c)** $\int \arctan x\,dx$

d) $\int \sec 3x\,dx$ **e)** $\int \frac{x}{25+x^2}\,dx$ **f)** $\int \frac{x}{1+25x^4}\,dx$ **g)** $\int \sin^2 2\pi x\,dx$

5. a) Determine $\int(\ln x)^2\,dx$.

b) Let R be the region enclosed by $y = \ln x$, the x-axis, and $x = e$ in the first quadrant. Rotate R about the x-axis and find the volume.

6. Let R be the region enclosed by $y = \cos x$, the x-axis, $x = 0$, and $x = \pi/2$ in the first quadrant. Rotate R about the y-axis. Find the volume of the resulting solid using shells. (Ans: $\pi^2 - 2\pi$)

7. Evaluate these limits.

a) $\lim_{x\to 0^+} 2x\ln x$ **b)** $\lim_{x\to\infty} \ln(2x+9) - \ln(x+7)$ **c)** $\lim_{x\to\infty}(1+\frac{2}{x})^x$ **d)** $\lim_{n\to\infty}\sqrt[n]{n}$ **e)** $\lim_{x\to\infty}(2x)^{1/x}$

8. Try the following.

a) $\int \frac{4}{\sqrt{4+x^2}}\,dx$ **b)** $\int \frac{4x}{(4+x^2)^{3/2}}\,dx$ **c)** $\int \frac{4x^2}{\sqrt{4-x^2}}\,dx$

d) $\int \frac{4x+1}{x^2-5x+4}\,dx$ **e)** $\int \frac{4x+8}{x^2+4x+5}\,dx$ **f)** $\int \frac{4x^2+8x+2}{x(x+1)^2}$

9. a) Does the series $\sum_{n=1}^{\infty}\frac{1}{n^2+5n+6}$ converge?

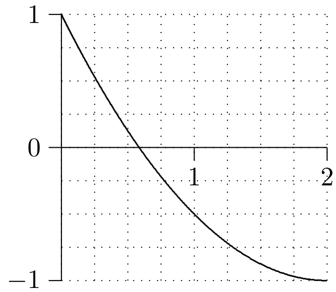
b) Do it again using another test.

c) List two other tests that could also be used.

10. Know the summation formula for $\sum_{k=1}^n k$ and $\sum_{k=1}^n k^2$

11. From a previous Final Exam: Draw and then estimate Right(4) for the graph of f on $[0, 2]$ below. Be careful about how you draw your rectangles. Watch the scale. How many rectangles should you draw?

a) Is your estimate an over-estimate or an under-estimate? Explain why.



12. Suppose that $f(x) = x^3 - 2x$ on $[0, 2]$.

a) Compute Right(n) for this situation.

b) Use your Riemann sum to find $\int_0^2 x^3 - 2x\,dx$. Then check your answer by using antidifferentiation.

13. If $f(x) = \int_x^2 e^{\sin t}\,dt$, what is $f'(x)$?

14. On the final exam for Math 131, Jody says that $\int \cos(\sqrt{x})\,dx = 2\cos(\sqrt{x}) + 2\sqrt{x}\sin(\sqrt{x}) + c$. Determine whether she should receive credit for her answer. Explain. Hint: I do not expect that you can do this integration.

15. Find the area enclosed by the curves $y = x^2 + 2x$ and $y = x^3 - 4x$. Draw the figure. (Ans: $21\frac{1}{12}$)

16. Find the volume of the solid that results when the region enclosed by $y = \sqrt{x}$ and $y = x$ is revolved about the x-axis. (Answer: $\pi/6$)

17. A small canal buoy is formed by taking the region in the first quadrant bounded by the y-axis, the parabola $y = 2x^2$, and the line $y = 5 - 3x$ and rotating it about the y-axis. (Units are feet.) Find the volume of this buoy. (Answer: 2π cu ft.)

18. Find the average value of $f(x) = \dfrac{1}{\sqrt{4 - 36x^2}}$ on the interval $[0, \frac{1}{6}]$. (Ans: $\pi/6$)

19. Determine whether $\displaystyle\sum_{n=2}^{\infty} \frac{2n\cos(n\pi)}{n^2 - 1}$ converges absolutely, conditionally, or not at all.

20. Determine whether these series converge or diverge. Carefully justify your answer.

a) $\displaystyle\sum_{n=1}^{\infty} \sqrt[n]{n}$ **b)** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4n^2 + 2}{3n^2 + n + 6}\right)^n$ **c)** $\displaystyle\sum_{n=0}^{\infty} \frac{5^n}{n!}$ **d)** $\displaystyle\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^{n^2}$

21. Find the interval of convergence for these power series.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{nx^n}{5^{n+1}}$ **b)** $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{3n - 2}$ **c)** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n^2 + 1}$ **d)** $\displaystyle\sum_{n=0}^{\infty} (-1)^n n^2 x^n$ **e)** $\displaystyle\sum_{n=1}^{\infty} \frac{(2n + 1)x^{2n}}{n^2}$

22. a) A tank is formed by rotating the region between $y = x^2$, the y-axis and the line $y = 4$ in the first quadrant around the y axis. The oil in the tank has density 50 lbs/ft^3. Find the work done pumping the oil to the top of the tank if there is only 1 foot of oil in the tank.

b) Suppose the tank is empty and is **filled** from a hole in the bottom to a depth of 3 feet. Find the work done.

Math 131 Final Review Answers

1. **a)** $-\int_0^3 f(x)\,dx = -0.9$ **b)** $= \int_{-1}^0 f(x)\,dx + \int_0^2 f(x)\,dx$ symmetry $= -\int_0^1 f(x)\,dx + \int_0^2 f(x)\,dx = -0.4 + 0.8 = .4$

c) $= \int_1^2 x\,dx + \int_1^2 f(x)\,dx = 1.5 + (0.8 - 0.4) = 1.9$ **d)** $\int_1^2 f(x-1)\,dx = \int_0^1 f(x)\,dx = 0.4$

2. $AL = \int_0^{\pi/3} \sqrt{1+\tan^2 x}\,dx = \int_0^{\pi/3} \sec x\,dx = \ln|\sec x + \tan x|\Big|_0^{\pi/3}$.

3. Use the alternating series test. Check the two conditions.

a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sqrt{n}}{n-1} \overset{\text{HP}}{=} \lim_{n\to\infty} \frac{n^{1/2}}{n^2} = \lim_{n\to\infty} \frac{1}{n^{3/2}} = 0$✓.

b) Decr? $f(x) = \frac{\sqrt{x}}{x-1}$. $f'(x) = \frac{\frac{x-1}{2\sqrt{x}} - \sqrt{x}}{(x-1)^2} = \frac{\frac{x-1-2x}{2\sqrt{x}}}{(x-1)^2} = \frac{\frac{-1-x}{\sqrt{x}}}{(x-1)^2} < 0$. So the terms are decreasing.✓

4. Brief Answers to the Mix-Up Problem: (All "$+c$".)

a) $e^{\tan x}$ **b)** $x\tan x - \ln|\sec x|$ **c)** $x\arctan x - \frac{1}{2}\ln|x^2+1|$
d) $\frac{1}{3}\ln|\sec 3x + \tan 3x|$ **e)** $\frac{1}{2}\ln(25+x^2)$ **f)** $\frac{1}{10}\arctan 5x^2$ **g)** $\frac{x}{2} - \frac{1}{8\pi}\sin 4\pi x$

5. **a)** Use parts twice, the first time $u = (\ln x)^2$ and the second time $u = 2\ln x$. Ans: $x(\ln x)^2 - 2x\ln x + 2x + c$.

b) From (a): $\int_1^e \pi(\ln x)^2\,dx = \pi\left[x(\ln x)^2 - 2x\ln x + 2x\right]\Big|_1^e = \pi(e-2)$.

6. Use parts with $u = x$. $V = \int_0^{\pi/2} 2\pi x\cos x\,dx = 2\pi[x\sin x - \int_0^{\pi/2}\sin x\,dx] = 2\pi[x\sin x + \cos x]\Big|_0^{\pi/2}$.

7. **a)** 0 **b)** $\ln 2$ **c)** e^2 **d)** 1 **e)** 1

8. **a)** Triangles: $x = 2\tan\theta$: $\int \frac{4}{\sqrt{4+x^2}}\,dx = \int \frac{4}{2\sec^2\theta} 2\sec^2\theta\,d\theta = \int 4\sec\theta\,d\theta = 4\ln|\sec\theta + \tan\theta| + c = 4\ln\left|\frac{\sqrt{4+x^2}+x}{2}\right| + c$

b) With triangles: $\int \frac{8\tan\theta}{8\sec^3\theta} 2\sec^2\theta\,d\theta = \int 2\sin\theta\,d\theta = -2\cos\theta + c = -\frac{4}{\sqrt{4+x^2}}$

b) Much easier by u-sub: $= \int 2u^{-3/2}\,du = \int -4u^{-1/2} + c = -4(4+x^2)^{-1/2} + c$.

c) $\int \frac{16\sin^2\theta}{2\cos\theta} 2\cos\theta\,d\theta = \int 16\sin^2\theta\,d\theta = -8\sin\theta\cos\theta + 8\theta + c = -2x\sqrt{4+x^2} + 8\arcsin(\frac{x}{2}) + c$

d) $\int \frac{17/3}{x-4} - \frac{5/3}{x-1}\,dx = \frac{17}{3}\ln|x-4| - \frac{5}{3}\ln|x-1| + c$.

e) $\int \frac{4x+8}{x^2+4x+5}\,dx = 2\ln|x^2+4x+5| + c$. (Use u-substitution.)

f) $\int \frac{4x^2+8x+2}{x(x+1)^2}\,dx = \int \frac{2}{x} + \frac{2}{x+1} + \frac{2}{(x+1)^2}\,dx = 2\ln|x| + 2\ln|x+1| - 2(x+1)^{-1} + c$.

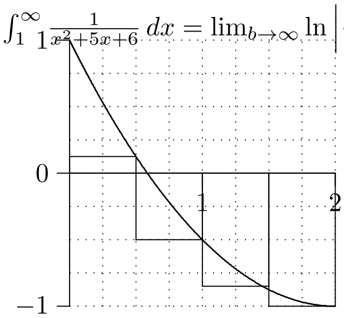
9. **a)** Since $\frac{1}{n^2+5n+6} < \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges by p-series test ($p = 2 > 1$), so $\sum \frac{1}{n^2+5n+6}$ by Direct Comparison.

b) Use the telescoping sum test (partial fractions): $\sum_{n=1}^{\infty} \frac{1}{n^2+5n+6} = \sum_{n=1}^{\infty} \frac{1}{n+2} - \frac{1}{n+3}$. Show that $s_n = \frac{1}{3} - \frac{1}{n+3}$. So $\lim_{n\to\infty} s_n = \frac{1}{3}$ and so the series converges to $\frac{1}{3}$.

c) Use LCT with $\sum \frac{1}{n^2}$ or use the integral test with partial fractions: $\int_1^{\infty} \frac{1}{x^2+5x+6}\,dx = \lim_{b\to\infty} \ln\left|\frac{x+2}{x+3}\right|\Big|_1^b = -\ln|\frac{3}{4}|$.

10. Look them up.

11. Right(4) ≈ -1.1 and it is an underestimate because f is decreasing.



12. Switch order of limits: then $f'(x) = -e^{\sin x}$ by the Second Fundamental Theorem of Calculus.

13. Use substitution with $u = \ln t$ and $du = \frac{1}{t}dt$. $x = e \Rightarrow u = 1$ and $x = e^2 \Rightarrow u = 2$. So $\int_1^2 \frac{1}{u}\,du = \ln|u|\Big|_1^2$.

14. Jody gets credit. Check that her answer is correct by taking the dervative.

15. Use $\int_{-1}^0 (x^3 - 4x) - (x^2 + 2x)\,dx + \int_0^3 (x^2 + 2x) - (x^3 - 4x)\,dx$

16. Use $\int_0^1 \pi(x-x^2)\,dx$.

17. $V = \int_0^1 2\pi x[(5-3x)-2x^2]\,dx$.

18. $AV = \frac{1}{1/6-0}\int_0^{1/6}\frac{1}{\sqrt{4-36x^2}}\,dx = \frac{1}{1/6}\int_0^{1/6}\frac{1}{\sqrt{a^2-u^2}}\,dx = \frac{1}{6}\cdot\frac{1}{6}\arcsin\frac{6x}{2}\Big|_0^{1/6} = \frac{1}{36}[\arcsin(1/2)-\arcsin(0)] = \pi/36$. Note: $u = 6x$ here so $du = 6dx$ or $\frac{1}{6}du = dx$.

19. Use $\cos(n\pi) = (-1)^n$. First check absolute convergence. $\sum_{n=2}^{\infty}\left|\frac{(-1)^n 2n}{n^2-1}\right| = \sum_{n=2}^{\infty}\frac{2n}{n^2-1}$. Notice $\frac{2n}{n^2-1} \approx \frac{1}{n}$. Use the limit comparison test. The terms of both series are positive and $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{2n}{n^2-1}\cdot\frac{n}{1} \overset{\text{HPwrs}}{=} \lim_{n\to\infty}\frac{2n^2}{2n^2} = 2$. Since the harmonic series $\sum_{n=2}^{\infty}\frac{1}{n}$ diverges (p-series with $p=1$), then $\sum_{n=2}^{\infty}\left|\frac{(-1)^n 2n}{n^2-1}\right|$ diverges by the limit comparison test. So the series does not converge absolutely. Check for conditional convergence using the alternating series test. The terms $a_n = \frac{2n}{\sqrt{n^2-1}}$ are positive. Check the two conditions. (i): $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{2n}{n^2-1} \overset{\text{HPwrs}}{==} \lim_{n\to\infty}\frac{2n}{n^2} = 0$.✓(ii): Non-increasing? $f(x) = \frac{2x}{x^2-1}$. So $f'(x) = \frac{2(x^2-1)-4x^2}{(x^2-1)^2} = \frac{-2-2x^2}{(x^2-1)^2} < 0$. So the terms are decreasing.✓By the alternating series test $\sum_{n=2}^{\infty}\frac{(-1)^n 2n}{n^2-1}$ converges so it is *conditionally convergent*.

20. These are brief answers. You should show all work involved. (I am trying to fit the answers on one page.)

a) Diverges using the nth term test.

b) Converges using the alternating series test.

c) Diverges by the root test ($r = 4/3 > 1$).

d) Converges by the ratio test ($r = 0 < 1$).

e) Converges by the root test ($r = \frac{1}{e} < 1$).

21. In each of these use the ratio test to find the radius of convergence R and then check the two endpoints.

a) After simplifying $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)x}{5n}\right| = \left|\frac{x}{5}\right| < 1 \iff |x| < 5$. $R = 5$. Check endpoints. At $x = 5$: $\sum_{n=0}^{\infty}\frac{n5^n}{5^{n+1}} = \sum_{n=0}^{\infty}\frac{n}{5}$ diverges by the n-th term test since $\lim_{n\to\infty}\frac{n}{5} = \infty \neq 0$. At $x = -5$: $\sum_{n=0}^{\infty}\frac{n(-5)^n}{5^{n+1}} = \sum_{n=0}^{\infty}\frac{(-1)^n n}{5}$ diverges by the n-th term test since $\lim_{n\to\infty}\frac{(-1)^n n}{5} \neq 0$. Interval: $(-5,5)$.

b) After simplifying $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(3n-2)x}{3n+1}\right| = |x| < 1$. $R = 1$. Check the endpoints. At $x = 1$: $\sum_{n=0}^{\infty}\frac{1}{3n-2}$. Use LCT with $\sum\frac{1}{n}$. $L = \lim_{n\to\infty}\frac{1}{3n-2}\cdot\frac{n}{1} \overset{\text{HP}}{=} \frac{1}{3}$. Since $0 < L < \infty$ and since $\sum\frac{1}{n}$ diverges ($p = 1 \leq 1$), $\sum\frac{1}{3n-2}$ diverges by LCT. At $x = -1$: $\sum_{n=0}^{\infty}\frac{(-1)^n}{3n-2}$ use AST. Check 2 conditions: (1) $\lim_{n\to\infty}\frac{1}{3n-2} = 0$✓. (2) Notice $\frac{1}{3(n+1)-2} < \frac{1}{3n-2}$.✓So by AST, the series converges at $x = -1$. Interval: $[-1,1)$.

c) After simplifying $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{[(n+1)^2+1)x}{n^2+1}\right| = \lim_{n\to\infty}\left|\frac{(n^2+2n+2)x}{n^2+1}\right| = \lim_{n\to\infty}\left|\frac{(1+\frac{2}{n}+\frac{2}{n^2})x}{1+\frac{1}{n^2}}\right| = |x| < 1$. $R = 1$. Endpoints. At $x = 1$: $\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+1}$. Use AST. Check 2 conditions: (1) $\lim_{n\to\infty}\frac{1}{n^2+1} = 0$✓. (2) Notice $\frac{1}{(n+1)^2+1} < \frac{1}{n^2+1}$.✓So by AST, the series converges at $x = 1$. At $x = -1$: $\sum_{n=0}^{\infty}\frac{1}{n^2+1}$. Use DCT with $\sum\frac{1}{n^2}$. The terms of both are positive and $\frac{1}{n^2+1} < \frac{1}{n^2}$. Since $\sum\frac{1}{n^2}$ converges ($p = 2 > 1$), the series converges. Interval: $[-1,1]$.

d) After simplifying $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n^2)x}{(n+1)^2}\right| = \lim_{n\to\infty}\left|\frac{(n^2)x}{n^2+2n+1}\right| = \lim_{n\to\infty}\left|\frac{(1)x}{1+\frac{2}{n}+\frac{1}{n^2}}\right| = |x| < 1$. $R = 1$. Check the endpoints. At $x = 1$: $\sum_{n=0}^{\infty}(-1)^n n^2$ diverges by nth term test: $\lim_{n\to\infty}(-1)^n n^2\text{DNE} \neq 0$. At $x = -1$: $\sum_{n=0}^{\infty} n^2$ diverges by the n-th term test: $\lim_{n\to\infty} n^2\infty \neq 0$. Interval: $(-1,1)$.

e) After simplifying $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n^2(2n+3)x^2}{(2n+1)(n+1)^2}\right| \overset{\text{HPwrs}}{=} \lim_{n\to\infty}\left|\frac{2n^3x^2}{2n^3}\right| = |x^2| < 1$ so $|x| < 1$ and $R = 1$. Check the endpoints. At $x = 1$: $\sum_{n=1}^{\infty}\frac{2n+1}{n^2}$ use limit comparison with $\sum\frac{1}{n}$. $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{2n+1}{n^2}\cdot\frac{n}{1} = \lim_{n\to\infty}\frac{2n^2+n}{n^2} \overset{\text{HPwrs}}{=} 2$. Since $\sum\frac{1}{n}$ diverges (p-series, $p = 1$), then $\sum_{n=1}^{\infty}\frac{2n+1}{n^2}$ by limit comparison. At $x = -1$: $\sum_{n=1}^{\infty}\frac{2n+1(-1)^{2n}}{n^2} = \sum_{n=1}^{\infty}\frac{2n+1}{n^2}$ which again diverges. Interval: $(-1,1)$.

22. a) Work $= D\int_a^b A(y)[H-y]\,dy = 50\int_0^1 \pi(y)[4-y]\,dy = 50\pi\left(2y^2-\frac{y^3}{3}\right)\Big|_0^1 = 50\pi\left[\left(2-\frac{1}{3}\right)\right] = \frac{250\pi}{3}$ ft-lbs.

b) Work $= D\int_a^b A(y)[H-y]\,dy = 50\int_0^3 \pi(y)[y-0]\,dy = 50\pi\left(\frac{y^3}{3}\right)\Big|_0^3 = 50\pi\left[\left(2-\frac{1}{3}\right)\right] = 450\pi$ ft-lbs.