2.4 *An Application of Definite Integrals: Average Value*

Here is another simple example of an application of the definite integral which points out the power of the definition of the integral as a Riemann sum.

Suppose that we want to know the average temperature for February 27, 2015 in Geneva (see Figure 2.44). How might we find it? Well, we could take the $n = 24$ hourly temperature recordings, add them together, and then divide by 24 might as we might do to find any average. Is the average 19.7 as listed in the table? What 'average' is that?

Time	**Temp**
1:00	12
2:00	13
3:00	13
4:00	12
5:00	11
6:00	12
7:00	13
8:00	18
9:00	21
10:00	24
11:00	26
12:00	28
13:00	28
14:00	29
15:00	29
16:00	27
17:00	25
18:00	24
19:00	21
20:00	19
21:00	17
22:00	18
23:00	17
24:00	16
Ave	19.7

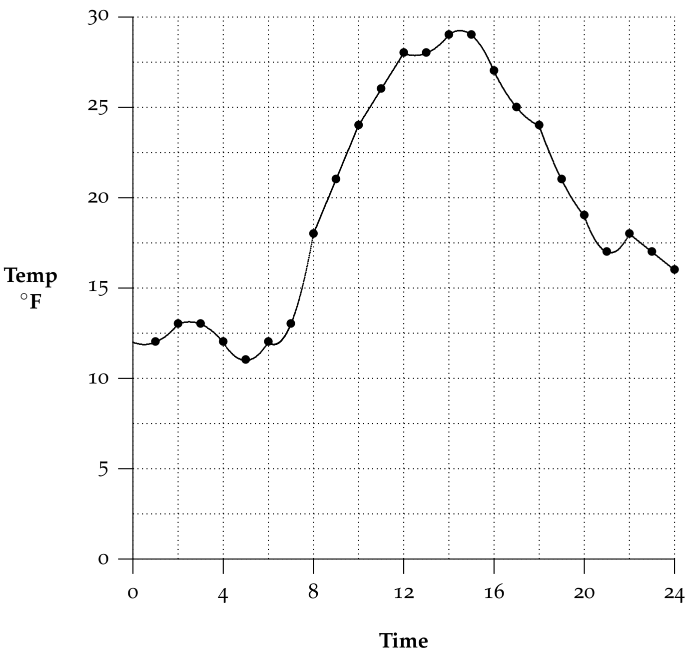

Figure 2.44: A graph of the temperature on February 27, 2003 using the data to the right.

The average of 19.7 'privileges' those recordings made on the hour. We could get a better estimate if we recorded temperatures every half-hour, or every 5 minutes, or every minute, or perhaps every second. The more recordings we use, the better the 'average.' Let's generalize the problem. In doing so, we are sometimes able to see the pattern which will help us solve the particular problem we are interested in.

The Average Value Problem: Let f be a continuous function on the closed interval $[a, b]$. Find the average value of f on $[a, b]$.

Solution. We make use of the outline of steps on page 35. But how do we subdivide an average and make it product? As usual, start by dividing $[a, b]$ into n equal subintervals with partition points $\{x_0, x_1, \ldots, x_n\}$. Then, as we suggested above,

$$\text{Average of } f \approx \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} = \sum_{k=1}^{n} f(x_k) \cdot \frac{1}{n}. \tag{2.10}$$

The summation looks almost like a Riemann sum except we now have $\frac{1}{n}$ instead of Δx. But hold on!

$$\Delta x = \frac{b-a}{n}$$

so

$$\frac{1}{n} = \frac{b-a}{n} \cdot \frac{1}{b-a} = \frac{\Delta x}{b-a}.$$

Substituting this back in equation (2.10) gives

$$\text{Average of } f \approx \sum_{k=1}^{n} f(x_k) \cdot \frac{\Delta x}{b-a} = \frac{1}{b-a} \sum_{k=1}^{n} f(x_k)\Delta x. \tag{2.11}$$

Now we do have a Riemann sum in (2.11). We have already remarked that if we let n increase (take more points in our average), we should get a more accurate approximation. The best approximation occurs when we take a limit as the number of evaluation points $n \to \infty$. In other words

$$\text{Average of } f = \lim_{n\to\infty} \frac{1}{b-a} \sum_{k=1}^{n} f(x_k)\Delta x = \frac{1}{b-a} \int_a^b f(x)\,dx. \tag{2.12}$$

We know this limit exists and equals the definite integral because f is continuous (see Theorem 1.4.2). Having carried out the steps on page 35, we are led to make the following definition.

DEFINITION 2.4.1 (Average Value). Assume that f is integrable on $[a,b]$. Then the **average value of f on** $[a,b]$ is denoted by $\overline{f} = f_{\text{ave}}$ and is defined by

$$\overline{f} = f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

EXAMPLE 2.4.2. Find the average value of $f(x) = \sqrt{x}$ on $[0,9]$. Then find a point c between 0 and 9 where $f(c) = \overline{f} = f_{\text{ave}}$.

Solution. Using Definition 2.4.1

$$\overline{f} = f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)\,dx = \frac{1}{9-0} \int_0^9 \sqrt{x}\,dx = \frac{1}{9} \cdot \frac{2}{3} x^{3/2} \Big|_0^3 = \frac{2}{27}(27-0) = 2.$$

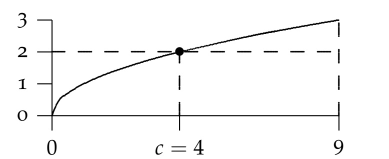

Figure 2.45: The average value of $\sqrt{x}$ on $[0,9]$ is 2. This value actually occurs at $c = 4$.

The average value is shown in Figure 2.45. Think of the original curve as a wave in a fish tank. The wave settles to the average value of the function. The area of the rectangle formed using the average value as the height is the same as the area under the original curve. Notice that there is a point, namely $c = 4$ at which the height of the curve is the same as the average value ($f(4) = 2$).

YOU TRY IT 2.16. Find the average value of $f(x) = x^2$ on $[-1,2]$. For which values c in the interval does the average value actually occur?

ANSWER TO YOU TRY IT 2.16. The average value is 1. It occurs at $c = \pm 1$.

EXAMPLE 2.4.3. A patient being treated for emphysema is tested with a spirometer to measure lung capacity. The data show the volume of air in the patient's lungs during inhalation is given by $V(t) = 1 - \cos\left(\frac{\pi t}{2}\right)$ pints over the time interval $[0,2]$ seconds. Find the average volume of air in the his lungs during this period.

Solution. Using Definition 2.4.1

$$\begin{aligned} \text{Average V} &= \frac{1}{2-0} \int_0^2 1 - \cos\left(\frac{\pi t}{2}\right)\,dt \\ &= \frac{1}{2}\left[t - \frac{2}{\pi}\sin\left(\frac{\pi t}{2}\right)\right]\Bigg|_0^2 = \frac{1}{2}[(2-0)-(0-0)] = 1 \text{ pint.} \end{aligned}$$

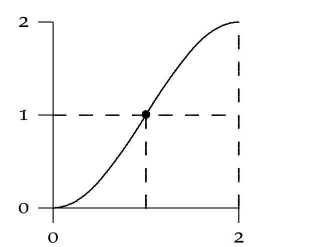

Figure 2.46: The average value of $V(t) = 1 - \cos\left(\frac{\pi t}{2}\right)$ on $[0,2]$ is 1.

In Figure 2.46 notice how the area under the curve above the average value balances out the missing area below the average value and the curve. Notice also that the average value actually occurs at $c = 1$ (since $f(1) = 1$).

YOU TRY IT 2.17. A patient being treated for pulmonary fibrosis is tested with a spirometer to measure lung capacity. The data show the volume of air in the patient's lungs during both the inhalation and exhalation cycles is given by

$$V(t) = 1 - \cos\left(\frac{2\pi t}{5}\right) \text{ pints}$$

over the time interval $[0, 5]$ seconds. Find the average volume of air in the his lungs during this period. At what time(s) does this volume occur?

ANSWER TO YOU TRY IT 2.17. 1 pint and it occurs at $c = \frac{5}{4}$ and $\frac{15}{4}$.

WEBWORK: Click to try Problems 47 through 49. Use GUEST login, if not in my course.

YOU TRY IT 2.18 (Hand in: Challenge). The phases of the moon occur in a regular, predictable 28 day lunar cycle. Data (for Boston) from *The Old Farmer's Almanac* indicate that the hours, $v(x)$, the moon is visible per day over the course of the cycle is $v(x) = 2.615\cos(\frac{\pi}{14}x - \pi) + 12.435$, where x is the day of the cycle.

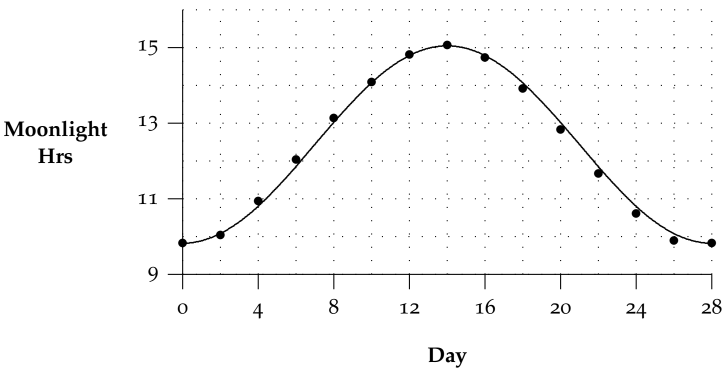

Figure 2.47: Data (for Boston) from *The Old Farmer's Almanac* for the number of hours the moon is visible per day during its 28-day cycle.

(*a*) Find the average number of hours the moon is visible per day in Boston using calculus

(*b*) Extra Credit: Use your brain to explain why you should have expected this answer!

In Example 2.4.2 and Example 2.4.3 we noted that there were points where the average value of the function actually occurred. This turns out to always be true as long as $f(x)$ is continuous.

THEOREM 2.4.4 (Mean Value Theorem for Integrals: **MVTI**). If f is continuous on $[a, b]$, then there's a point c in $[a, b]$ so that

$$\int_a^b f(t)\,dt = f(c) \cdot (b - a).$$

In other words

$$f(c) = \frac{1}{b-a}\int_a^b f(t)\,dt = \overline{f} = f_{\text{ave}}.$$

Proof. As usual, for x in $[a, b]$, define our accumulation function as

$$A(x) = \int_a^x f(t)\,dt.$$

Then by FTC I, $A'(x) = f(x)$. So $A(x)$ is differentiable on $[a, b]$ so it is continuous there. By the original MVT, there is a point c in $[a, b]$ so that

$$A'(c) = \frac{A(b) - A(a)}{b-a} = \frac{\int_a^b f(t)\,dt - \int_a^a f(t)\,dt}{b-a} = \frac{\int_a^b f(t)\,dt - 0}{b-a} = \frac{1}{b-a}\int_a^b f(t)\,dt.$$

But $A'(c) = f(c)$, so

$$f(c) = \frac{1}{b-a}\int_a^b f(t)\,dt = \overline{f} = f_{\text{ave}}.$$

□

EXAMPLE 2.4.5. Let $f(x) = 2\cos(x)$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Find the point c in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ where $f(c) = f_{ave}$.

Solution. First notice that $2\cos x$ is differentiable so it is continuous. So the MVTI applies. Next we need to determine $\overline{f} = f_{\text{ave}}$. Using Definition 2.4.1

$$\overline{f} = f_{\text{ave}} = \frac{1}{\pi/2 - (-\pi/2)}\int_{-\pi/2}^{\pi/2} 2\cos x\,dx = \frac{1}{\pi}(2\sin x)\Big|_{-\pi/2}^{\pi/2} = \frac{1}{\pi}[2-(-2)] = \frac{4}{\pi}.$$

So we need to find c in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so that $f(c) = \overline{f} = f_{\text{ave}} = \frac{4}{\pi}$. In other words,

$$\begin{aligned} f(c) = 2\cos c &= \frac{4}{\pi} \\ \cos c &= \frac{2}{\pi} \\ c &= \arccos\left(\frac{2}{\pi}\right) \approx \pm 0.8806892354. \end{aligned}$$

YOU TRY IT 2.19. Let $f(x) = x^2 - 1$ on $[0,3]$. Does the MVTI apply to this function? Why? If so, find the point(s) c in $[0,3]$ so that $f(c) = \overline{f} = f_{\text{ave}}$.

ANSWER TO YOU TRY IT 2.19. The MVTI does apply because f is a polynomial (continuous). $c = \sqrt{3}$.

EXAMPLE 2.4.6. Find the average value of $f(x) = |x-3|$ on $[0,5]$ and determine the points c where the average occurs.

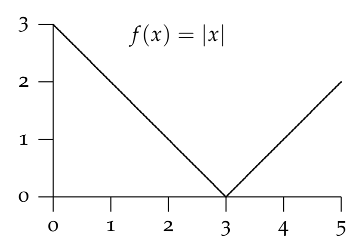

Figure 2.48: The area under $f(x) = |x-3|$ on $[0,5]$ consists of triangles.

Solution. By Definition 2.4.1,

$$\begin{aligned} \overline{f} = f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx &= \frac{1}{5-0}\int_0^5 |x-3|\,dx \\ &\overset{\text{See graph}}{=} \frac{1}{5}\left[\frac{1}{2}(3)(3) + \frac{1}{2}(2)(2)\right] \\ &= 1.3. \end{aligned}$$

Note that we were able to easily evaluate the integral by using the geometry of the the two triangles.

Since $|x-3|$ is continuous, the MVTI says there's a point c in $[0,5]$ where $f(c) = \overline{f} = f_{\text{ave}}$. We need

$$|c-3| = 1.3 \iff \begin{cases} c-3 = 1.3 \\ c-3 = -1.3 \end{cases} \iff \begin{cases} c = 4.3 \\ c = 1.7 \end{cases}.$$

Both points are in the interval.

2.5 *Applications of Riemann Sums and the FTC: Net Distance Travelled*

Suppose we know that the velocity of an object traveling along a line (think car on a straight highway) is given by a continuous function $v(t)$, where t represents time on the interval $[a, b]$. How might we determine the net distance the object has travelled? Well, we know that if the velocity were *constant*, then

$$\text{distance} = \text{rate} \times \text{time}.$$

Observe: Distance has been expressed as product, much the way we assumed earlier that the area of a rectangle could be expressed as a product:

$$\text{area of a rectangle} = \text{height} \times \text{base}.$$

☞ When first taking calculus it is easy to confuse the integration process (with its Riemann sums) with simple 'antidifferentiation.' While the First Fundamental Theorem connects these two, they are not the same thing. Most important, though, the determination of many quantities can be approximated (interpreted) as Riemann sums and hence evaluated as definite integrals *even though* it is not obvious at the outset that antidifferentiation should be involved. The Riemann sum part turns out to be critical.

We can extend this analogy to Riemann sums and area under curves. While the velocity is *not constant on long intervals* since the velocity is continuous it is *nearly constant on short time intervals*. So divide the time interval using a regular partition $\{t_o, t_1, t_2, \ldots, t_n\}$ of n subintervals of length Δt. Next, pick any point in the kth subinterval (we might as well choose the right-hand endpoint t_k for convenience) and evaluate the velocity $v(t_k)$ there. Then the distance traveled during the kth time interval approximated as

$$\text{distance} = \text{rate} \times \text{time} \approx v(t_k) \times \Delta t.$$

Since the net distance travelled is the sum of the distances traveled on each subinterval which is approximately

$$\text{Net Distance} \approx \sum_{k=1}^{n} v(t_k) \times \Delta t.$$

The approximation is improved by letting n get large and taking a limit.

$$\boxed{\text{Net Distance} = \lim_{n\to\infty} \sum_{k=1}^{n} v(t_k) \times \Delta t = \int_a^b v(t)\, dt.} \tag{2.13}$$

Since v was assumed to be continuous, then we know that the limit exists and can be evaluated as a definite integral using antidifferentiation assuming we know an appropriate antiderivative. Finally, think about how we interpreted definite integrals geometrically: as (net) area under a curve. What we have just shown is that the net distance travelled over the time interval $[a, b]$ is just the *net area under the velocity curve*. That's not obvious at first.

What's your point? The key point here is that we were able to use a 'divide and conquer' process to determine the displacement. Let's list it as a series of steps.

- We subdivided the quantity into small bits,
- and we were able to approximate the each bit as a *product*.
- When we reassembled (summed) the bits, we found we had a Riemann sum.
- Once we had a Riemann sum we take a limit as the number of bits got large.
- The limit was a definite integral
- which we could evaluate easily (if we know an antiderivative) using the Fundamental Theorem of Calculus.

We will use this process repeatedly over the next few weeks.