Improper Integrals

Introduction

When we defined the definite integral

$$\int_a^b f(x)\,dx$$

we assumed that f was *continuous* on $[a,b]$ where $[a,b]$ was a *finite*, closed interval. There are at least two ways this definition can fail to be satisfied:

1. The interval $[a,b]$ may be *infinite* with either $a=-\infty$ or $b=+\infty$ (or both), e.g.,

$$\int_1^\infty \frac{1}{x^2}\,dx.$$

2. The function may have one or more *infinite discontinuities* (vertical asymptotes) somewhere on $[a,b]$, e.g., in the integral

$$\int_{-1}^1 \frac{1}{x}\,dx$$

the integrand $\frac{1}{x}$ has infinite an discontinuity at 0.

Such integrals are called **improper integrals** because they do not satisfy the definition of the definite integral. Nonetheless, we will see that under certain conditions we can make sense numerically of such improper integrals. We will discuss each type of violation separately.

Type 1: Improper Integrals on Infinite Intervals

We start with a definition that describes how we can make sense of definite integrals on infinite intervals. The definition is divided into three parts depending on the type of interval.

DEFINITION 7.6.1 (Improper Integrals on Infinite Intervals)**.** There are three types of infinite intervals to consider.

(*a*) Intervals of the form $[a,\infty)$: If f is continuous on $[a,\infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

provided the limit exists. If limit exists, we say the improper integral **converges**. Otherwise it **diverges**.

(*b*) Intervals of the form $(-\infty,b]$: If If f is continuous on $(-\infty,b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$

provided the limit exists.

(c) Intervals of the form $(-\infty, \infty)$: If f is continuous on $(-\infty, \infty)$, then we define

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty}\int_a^c f(x)\,dx + \lim_{b\to\infty}\int_c^b f(x)\,dx,$$

provided both limits exist. Here c can be any convenient real number—often 0.

EXAMPLE 7.6.2. Let's start with an easy example. Is $\int_1^{\infty} \frac{1}{x^2}\,dx$ convergent?

Solution. Using Definition 7.6.1, we evaluate the appropriate limit:

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left(-\frac{1}{x}\Big|_1^b\right) = \lim_{b\to\infty}\left(-\frac{1}{b}+1\right) = 1.$$

The improper integral converges to 1. Geometrically, this means that the area under the curve $f(x) = \frac{1}{x^2}$ on the infinitely long interval $[1, \infty)$ is 1 square unit (see Figure 7.1.)

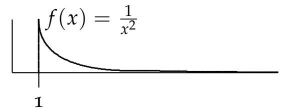

Figure 7.1: The area under the curve $f(x) = \frac{1}{x^2}$ on the infinitely long interval $[1, \infty)$ is 1 square unit.

EXAMPLE 7.6.3. Here's a similar example. Is $\int_1^{\infty} \frac{1}{x}\,dx$ convergent?

Solution. The graph of this function (see Figure 7.2) is remarkably similar to the previous example. Using Definition 7.6.1, we evaluate the appropriate limit:

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}\left(\ln|x|\Big|_1^b\right) = \lim_{b\to\infty}[\ln b - 0] = +\infty.$$

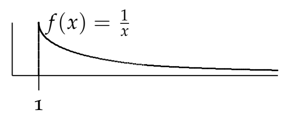

Figure 7.2: The area under the curve $f(x) = \frac{1}{x}$ on the infinitely long interval $[1, \infty)$ is unbounded or infinite, despite the fact that the graph seems similar to that of $\frac{1}{x^2}$.

The natural log function increases without bound as x increases so the limit does not exist. We say that the improper integral *diverges* or does not exist. Geometrically, this means that the area under the curve $f(x) = \frac{1}{x}$ on the infinitely long interval $[1, \infty)$ is unbounded as x increases.

EXAMPLE 7.6.4. Here's a more interesting example. Is $\int_1^{\infty} \frac{-2}{x^2+x}\,dx$ convergent?

Solution. First notice that this function is continuous on the interval $[1, \infty)$ since the denominator is 0 only at $x = -1$ and 0. Use partial fractions for the integration.

$$\frac{-2}{x^2+x} = \frac{A}{x} + \frac{B}{x+1} = \frac{Ax + A + Bx}{x^2+x}.$$

Comparing the numerators

$$\begin{array}{lrcll} x\text{'s:} & 0 & = & A + B & \\ \text{constants:} & -2 & = & A & \Rightarrow B = 2 \end{array}$$

Using Definition 7.6.1, we evaluate the appropriate limit:

$$\begin{aligned}
\int_1^{\infty} \frac{-2}{x^2+x}\,dx = \lim_{b\to\infty}\int_1^b \left(-\frac{2}{x}+\frac{2}{x+1}\right)dx &= \lim_{b\to\infty}\left(-2\ln|x| + 2\ln|x+1|\Big|_1^b\right) \\
&= \ln\left(2\lim_{b\to\infty}\left|\frac{x+1}{x}\right|\Big|_1^b\right) \\
&= \lim_{b\to\infty}\left(2\ln\left|\frac{b+1}{b}\right| - 2\ln 2\right) \\
&= 2\ln\lim_{b\to\infty}\left(\left|\frac{b+1}{b}\right|\right) - 2\ln 2 \\
&\stackrel{\text{l'Ho}}{=} \lim_{b\to\infty} 2\ln 1 - 2\ln 2 \\
&= -2\ln 2
\end{aligned}$$

So the improper integral converges. Notice how we simplified the natural log expression first using log rules to change a difference of logs into a log of a quotient.

EXAMPLE 7.6.5. Does $\int_{-\infty}^{0} \frac{1}{\sqrt{4-x}}\,dx$ exist (converge)?

Solution. This time we apply Definition 7.6.1 (part b) and use a mini-substitution to do the integral).[1]

[1] Use $u = 4 - x$ and $-du = dx$.

$$\int_{-\infty}^{0} \frac{1}{\sqrt{4-x}}\,dx = \lim_{a\to-\infty} \int_{a}^{0} (4-x)^{-1/2}\,dx = \lim_{a\to-\infty} -2(4-x)^{1/2}\Big|_{a}^{0}$$
$$= \lim_{a\to-\infty} \left[-2[2-(4-a)^{1/2}]\right] = +\infty.$$

The integral diverges.

Double Trouble

Some improper integrals have infinite upper and lower limits. According to Definition 7.6.1 we must break the integral into two pieces and evaluate each separately. Only if both integrals converge does the entire integral converge.

EXAMPLE 7.6.6. Does $\int_{-\infty}^{\infty} \frac{1}{1+4x^2}\,dx$ converge?

Solution. First we split the integral into two pieces, say at $a = 0$. Then using Definition 7.6.1 we have

$$\int_{-\infty}^{\infty} \frac{1}{1+4x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+4x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+4x^2}\,dx$$

Next we must evaluate each improper integral. You should recognize the integrand as leading to an inverse tangent function.[2]

[2] Hint: With more complicated improper integrals, it often makes sense to determine the antiderivative first. Then do the evaluation with the appropriate limits. In this case recall that

$$\boxed{\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\arctan\frac{u}{a} + c.}$$

In this example, $a = 1$ and $u = 2x$, so $\frac{1}{2}du = dx$.

$$\int_{-\infty}^{0} \frac{1}{1+4x^2}\,dx = \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{1+4x^2}\,dx = \lim_{a\to-\infty} \tfrac{1}{2}\arctan 2x\Big|_{a}^{0}$$
$$= \lim_{a\to-\infty} \tfrac{1}{2}(\arctan 0 - \arctan 2a)$$
$$= \frac{1}{2}\left[0 - \left(-\frac{\pi}{2}\right)\right]$$
$$= \frac{\pi}{4}.$$

Similarly, the other piece of the integral is

$$\int_{0}^{\infty} \frac{1}{1+4x^2}\,dx = \lim_{b\to\infty} \int_{0}^{b} \frac{1}{1+4x^2}\,dx = \lim_{b\to\infty} \tfrac{1}{2}\arctan 2x\Big|_{0}^{b}$$
$$= \lim_{a\to-\infty} \tfrac{1}{2}(\arctan 2a - \arctan 0)$$
$$= \frac{1}{2}\left[\frac{\pi}{2} - 0\right]$$
$$= \frac{\pi}{4}.$$

Putting this all together,

$$\int_{-\infty}^{\infty} \frac{1}{1+4x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+4x^2}\,dx + \int_{-\infty}^{\infty} \frac{1}{1+4x^2}\,dx = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

The integral converges.

Already these examples illustrate the importance of being able to evaluate limits at infinity with confidence.

EXAMPLE 7.6.7. Here's a similar problem. Does $\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx$ converge?

Solution. We split the integral into two pieces at $a = 0$.

$$\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{x}{1+x^2}\,dx + \int_{0}^{\infty} \frac{x}{1+x^2}\,dx$$

Next we must evaluate each improper integral. You should recognize this as a substitution integral.[3]

$$\int_{-\infty}^{0} \frac{x}{1+x^2}\,dx = \lim_{a\to-\infty} \int_{a}^{0} \frac{x}{1+x^2}\,dx = \lim_{a\to-\infty} \tfrac{1}{2}\ln(1+x^2)\Big|_a^0$$
$$= \lim_{a\to-\infty} \tfrac{1}{2}[\ln 1 - \ln(1+a^2)] = -\infty.$$

[3] $u = 1+x^2$ and $\frac{1}{2}\,du = x\,dx$. So $\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln(1+x^2)+c$.

Since this piece of the total integral diverges, by Definition 7.6.1 the entire improper integral diverges. There is no need to try to evaluate the second piece.

Remember, $\lim_{x\to\infty} \ln x = +\infty$.

EXAMPLE 7.6.8. Does $\int_{-\infty}^{0} xe^x\,dx$ converge?

Solution. This is an integration by parts problem. Let $u = x$ so $du = dx$ and then $dv = e^x\,dx$ so $v = e^x$. Then

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x = (x-1)e^x.$$

Returning to the improper integral

$$\int_{-\infty}^{0} xe^x\,dx = \lim_{a\to-\infty} \int_{a}^{0} xe^x\,dx = \lim_{a\to-\infty} (x-1)e^x\Big|_a^0 = \lim_{a\to-\infty} [-1-(a-1)e^a].$$

Notice that $\lim_{a\to-\infty} (a-1)e^{-a}$ has the form $-\infty\cdot 0$ so we put it into $-\frac{\infty}{\infty}$ form and use l'Hôpital's rule.

$$\lim_{a\to-\infty} (a-1)e^a = \lim_{a\to-\infty} \frac{(a-1)}{e^{-a}} \stackrel{\text{l'Ho}}{=} \lim_{a\to-\infty} \frac{1}{\underbrace{-e^{-a}}_{\searrow -\infty}} = 0.$$

Putting all our work together,

$$\int_{-\infty}^{0} xe^x\,dx = \lim_{a\to-\infty} [-1-(a-1)e^{-a}] = -1-0 = -1.$$

The integral converges.

YOU TRY IT 7.1. Determine which of these integrals converge. If the integral converges, determine its value.

(a) $\int_{e}^{\infty} \frac{1}{x(\ln x)^5}\,dx$ (b) $\int_{2}^{\infty} \frac{4}{x^2-1}\,dx$ (c) $\int_{2}^{\infty} \frac{4x}{x^2-1}\,dx$ (d) $\int_{-1}^{\infty} \frac{x}{(x^2+2)^3}\,dx$

ANSWER TO YOU TRY IT 7.1. NOT in order: $\frac{1}{36}$, diverges, $2\ln 3$, and $\frac{1}{4}$.

WEBWORK: Click to try Problems 125 through 127. Use GUEST login, if not in my course.

YOU TRY IT 7.2. Prove that if $n > 1$, then $\int_{1}^{\infty} \frac{1}{x^n}\,dx$ converges to $\frac{1}{n-1}$.

YOU TRY IT 7.3. Prove that if $n < 1$, then $\int_{1}^{\infty} \frac{1}{x^n}\,dx$ diverges. Note: When n is negative, remember that $\int_{1}^{\infty} \frac{1}{x^n}\,dx = \int_{1}^{\infty} x^{-n}\,dx$ where $-n$ is now a positive number.

Putting the last two exercises together with Example 7.6.3 you have proven the following result:

THEOREM 7.6.9 (The p-Power Test). For any real number p

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1}, & \text{if } p > 1 \\ \text{diverges}, & \text{if } p \le 1 \end{cases}.$$

YOU TRY IT 7.4. Evaluate each of these integrals or determine that it diverges by using Theorem 7.6.9. You should not need to do any antidifferentiation.

(a) $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ (b) $\displaystyle\int_1^\infty \frac{1}{x^{2/3}}\,dx$ (c) $\displaystyle\int_1^\infty \frac{2}{x^7}\,dx$ (d) $\displaystyle\int_1^\infty \frac{1}{x^{-15}}\,dx$

YOU TRY IT 7.5 (Gabriel's Horn). The infinitely-long solid formed by revolving the curve $f(x) = \frac{1}{x}$ about the x-axis over the interval $[1, \infty)$ is called **Gabriel's Horn**. (Do you know why?) Show that the volume is finite. See Figure 7.3

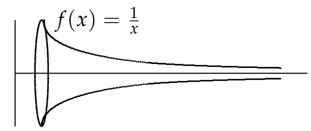

Figure 7.3: The area under the curve $f(x) = \frac{1}{x}$ is rotated around the x-axis.

Problems

1. Determine $\displaystyle\int_2^\infty \frac{9}{x^2 + x - 2}\,dx$.

2. **Bonus**: First determine $\displaystyle\int e^{-x}\cos x\,dx$. Then use your answer to determine $\displaystyle\int_0^\infty e^{-x}\cos x\,dx$. Show your work.