Type 2: Improper Integrals with Infinite Discontinuities

A second way that function can fail to be integrable in the ordinary sense is that it may have an infinite discontinuity (vertical asymptote) at some point in the interval. The simplest cases are when the function has such a discontinuity at an endpoint of the interval. An obvious example is

$$\int_0^4 \frac{1}{x^2}\, dx.$$

The integrand is not defined at $x = 0$ and, in fact, the function has a vertical asymptote there (see Figure 7.4).

As with integrals on infinite intervals, limits come to the rescue and allow us to define a second type of improper integral.
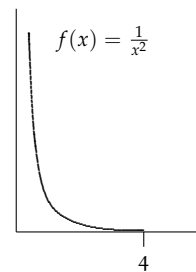
DEFINITION 7.6.10 (Improper Integrals with Infinite Discontinuities). Consider the following three types of infinite discontinuities.

(a) If f is continuous on $(a, b]$ and $\lim_{x \to a^+} f(x) = \pm\infty$, then

$$\int_a^b f(x)\, dx = \lim_{c \to a^+} \int_c^b f(x)\, dx$$

provided the limit exists. If limit exists, we say the improper integral **converges**. Otherwise it **diverges**.

(b) If f is continuous on $[a, b)$ and $\lim_{x \to b^-} f(x) = \pm\infty$, then

$$\int_a^b f(x)\, dx = \lim_{c \to b^-} \int_a^c f(x)\, dx$$

provided the limit exists.

(c) If f is continuous on $[a, b]$ *except* at some point p between a and b where it has an infinite discontinuity, then we define

$$\int_a^b f(x)\, dx = \int_a^p f(x)\, dx + \int_p^c f(x)\, dx,$$

provided both integrals converge.

Figure 7.4: The integral $f(x) = \frac{1}{x^2}$ on the interval $[0, 4]$ is improper because $f(x)$ has a vertical asymptote at $x = 0$.

Three Simple Examples

EXAMPLE 7.6.11. Does $\displaystyle\int_0^4 \frac{1}{x^2}\, dx$ exist (converge)?

Solution. As we saw above, this function has a vertical asymptote at $x = 0$, but is otherwise continuous on the interval $(0, 4]$. So we apply Definition 7.6.10.

$$\int_0^4 \frac{1}{x^2}\, dx = \lim_{c \to 0^+} \int_c^4 \frac{1}{x^2}\, dx = \lim_{c \to 0^+} -\frac{1}{x}\Big|_c^4 = \lim_{c \to 0^+} \left[-\frac{1}{4} + \frac{1}{c} \right] = +\infty.$$

The integral diverges.

EXAMPLE 7.6.12. Does $\displaystyle\int_0^4 \frac{1}{\sqrt{4 - x}}\, dx$ exist (converge)?

Solution. This time the function has a vertical asymptote at $x = 4$, but is otherwise continuous on the interval $[0, 4)$. So we apply Definition 7.6.10 (and use a mini-substitution to do the integral).[4]

[4] Use $u = 4 - x$ and $-du = dx$.

$$\int_0^4 \frac{1}{\sqrt{4-x}}\,dx = \lim_{c\to 4^-} \int_0^c (4-x)^{-1/2}\,dx = \lim_{c\to 4^-} -2(4-x)^{1/2}\Big|_0^c$$

$$= \lim_{c\to 4^-} \left[-2(4-c)^{1/2}+4\right] = 0+4 = 4.$$

The integral converges.

EXAMPLE 7.6.13. Does $\displaystyle\int_0^{\pi/2} \tan x\,dx$ exist (converge)?

Solution. Be careful. Remember that the tangent function has a vertical asymptote at $x = \frac{\pi}{2}$, but is otherwise continuous on the interval $[0, \frac{\pi}{2})$. So

$$\int_0^{\pi/2} \tan x\,dx = \lim_{c\to\frac{\pi}{2}^-} \int_0^c \tan x\,dx = \lim_{c\to\frac{\pi}{2}^-} \ln|\sec x|\Big|_0^c$$

$$= \lim_{c\to\frac{\pi}{2}^-} \left[\ln|\sec(c)| - 1\right] = +\infty.$$

Remember the that secant function goes to $+\infty$ as $c \to \frac{\pi}{2}^-$. And since the natural log function increases without bound as $x \to +\infty$, the entire expression goes to $+\infty$. So the integral diverges.

Combined with Other Techniques

Of course improper integrals can occur when more complicated integrals arise.

EXAMPLE 7.6.14. Does $\displaystyle\int_3^9 \frac{1}{\sqrt{x^2-9}}\,dx$ exist (converge)?

Solution. This time the function has a vertical asymptote at $x = 3$, but is otherwise continuous on the interval $(3,9]$. So we will eventually apply Definition 7.6.10. But first it makes sense to do the integration without limits because this involves a triangle substitution.

$$x = 3\sec\theta$$
$$dx = 3\sec\theta\tan\theta\,d\theta$$
$$\sqrt{x^2-9} = 3\tan\theta$$
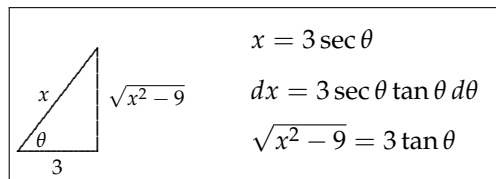
Now proceed with the substitution:

$$\int \frac{1}{\sqrt{x^2-9}}\,dx = \int \frac{1}{3\tan\theta}\cdot 3\sec\theta\tan\theta\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|\,d\theta$$

$$= \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right|.$$

So applying Definition 7.6.10

$$\int_3^9 \frac{1}{\sqrt{x^2-9}}\,dx = \lim_{c\to 3^+} \int_c^9 \frac{1}{\sqrt{x^2-9}}\,dx = \lim_{c\to 3^+} \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right|\Big|_c^9$$

$$= \lim_{c\to 3^+} \left[\ln\left|\frac{9}{3} + \frac{\sqrt{81-9}}{3}\right| - \ln\left|\frac{c}{3} + \frac{\sqrt{t^2-9}}{3}\right|\right]$$

$$= \ln(3+2\sqrt{2}) - \ln(1+0)$$

$$= \ln(3+2\sqrt{2}).$$

The integral converges. Whew!

EXAMPLE 7.6.15. Evaluate $\int_0^1 3x^2 \ln x \, dx$, if it converges.

Solution.   This time the function has a vertical asymptote at $x = 0$ where the natural log becomes undefined, but is otherwise continuous on the interval $(0,1]$. So we will eventually apply Definition 7.6.10. But first it makes sense to do the integration without limits because this involves an integration by parts.

Let $u = \ln x$ so $du = \frac{1}{x} dx$ and $dv = 3x^2 dx$, so $v = \int dv = \int 3x^2 dx = x^3$. Now proceed with the integration:

$$\int 3x^2 \ln x \, dx = x^3 \ln x - \int x^3 \cdot \frac{1}{x} dx = x^3 \ln x - \int x^2 dx = x^3 \ln x - \frac{x^3}{3}.$$

So applying Definition 7.6.10

$$\int_0^1 3x^2 \ln x \, dx = \lim_{c \to 0^+} \int_c^1 3x^2 \ln x \, dx = \lim_{c \to 0^+} x^3 \ln x - \frac{x^3}{3} \Big|_c^1$$

$$= \lim_{c \to 0^+} \left[\left(0 - \frac{1}{3} \right) - \left(t^3 \ln c - \frac{t}{3} \right) \right]$$

$$= \lim_{c \to 0^+} \left(\frac{c}{3} - \frac{1}{3} \right) - \lim_{c \to 0^+} \left(c^3 \ln c \right)$$

$$= -\frac{1}{3} - \lim_{c \to 0^+} \frac{\ln c}{\frac{1}{c^3}}$$

$$\overset{\text{l'Ho}}{=} -\frac{1}{3} - \lim_{c \to 0^+} \frac{\frac{1}{c}}{-\frac{3}{c^4}}$$

$$= -\frac{1}{3} + \lim_{c \to 0^+} \frac{c^3}{3}$$

$$= -\frac{1}{3}.$$

The integral converges. What fun!

EXAMPLE 7.6.16. What can you say about $\int_1^3 \frac{3}{x^2 - 3x} \, dx$?

Solution.   This time the function has a vertical asymptote at $x = 3$ where the $x^2 - 3x$ becomes undefined, but is otherwise continuous on the interval $[1,3)$. We will eventually apply Definition 7.6.10. It makes sense to do the integration (partial fractions) first without limits because this involves. The degree of the numerator is less than the degree of the denominator $(0 < 2)$ and the denominator factors into distinct linear factors: $x(x - 3)$.

$$\frac{3}{x^2 - 3x} = \frac{3}{x(x - 3)} = \frac{A}{x} + \frac{B}{x - 3} = \frac{Ax - 3A + Bx}{x^2 - 3x}.$$

Comparing the numerators of the first and last functions and solving for A and B gives

$$\begin{aligned} x\text{'s:} \quad & 0 = A + B \\ \text{constants:} \quad & 3 = -3A \quad \Rightarrow \quad A = -1, \quad B = 1 \end{aligned}$$

Consequently,

$$\int \frac{3}{x^2 - 3x} \, dx = \int \left(\frac{1}{x - 3} - \frac{1}{x} \right) dx = \ln |x - 3| - \ln |x| + c = \ln \left| \frac{x - 3}{x} \right| + c.$$

Notice that how we have simplified the answer! So applying Definition 7.6.10

$$\int_1^3 \frac{3}{x^2 - 3x}\, dx = \lim_{c \to 3^+} \int_1^c \frac{3}{x^2 - 3x}\, dx = \lim_{c \to 3^+} \ln\left|\frac{x-3}{x}\right|\,\Big|_1^c$$

$$= \lim_{c \to 3^+}\left[\ln\left|\frac{c-3}{c}\right|^{\nearrow \ln 0^+} - \ln\left|\frac{3-1}{1}\right|\right]$$

$$= -\infty.$$

The integral diverges.

Sometimes you need to be extra careful with improper integrals, as this next example illustrates.

EXAMPLE 7.6.17. What can you say about $\int_1^6 \frac{1}{x-4}\, dx$?

Solution. This time the function has a vertical asymptote at $x = 4$ where the denominator is 0, but is otherwise continuous on the intervals $[1, 4)$ and $(4, 6]$ (see Figure 7.5).

We must apply part (c) of Definition 7.6.10 and split the integral into two pieces at 4.

$$\int_1^6 \frac{1}{x-4}\, dx = \lim_{c \to 4^-} \int_1^c \frac{1}{x-4}\, dx + \lim_{c \to 4^+} \int_c^6 \frac{1}{x-4}\, dx.$$

We need to determine each integral separately

$$\lim_{c \to 4^-} \int_1^c \frac{1}{x-4}\, dx = \lim_{c \to 4^-} \ln|x - 4|\,\Big|_1^c = \lim_{c \to 4^-} \ln|c - 4| + \ln 3 = -\infty.$$

Consequently, the entire integral $\int_1^6 \frac{1}{x-4}\, dx$ diverges. We do not need to evaluate $\lim_{c \to 4^+} \int_c^6 \frac{1}{x-4}\, dx$.

So what was so hard about this? Well, it is easy to get this wrong because you might have missed the vertical asymptote at $x = 4$ and done the integration as follows:

$$\int_1^6 \frac{1}{x-4}\, dx = \ln|x - 4|\,\Big|_1^6 = \ln 2 - \ln 3 = \ln \tfrac{2}{3}.$$

If so, you would have missed the fact the integral was improper and actually *diverges*.

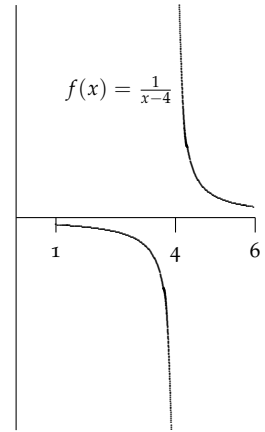WEBWORK: Click to try Problems 128 through 131. Use GUEST login, if not in my course.

Figure 7.5: $f(x) = \frac{1}{x-4}$ on the interval $[1, 6]$ is improper because $f(x)$ has a vertical asymptote at $x = 4$.

7.7   Problems

1. Determine these three integrals; for one use a theorem to make it quick.

(a) $\displaystyle\int_0^\infty \frac{x}{\sqrt{1+x^2}}\,dx$ (b) $\displaystyle\int_1^\infty \frac{1}{x^4}\,dx$ (c) $\displaystyle\int_0^1 \frac{2}{x^2-2x}\,dx$

2. Evaluate each improper integral; determine whether it converges or not. Reuse some part of your work in (a) for (f).

(a) $\displaystyle\int_2^\infty \frac{2}{x^2-1}\,dx$ (b) $\displaystyle\int_0^\infty \frac{2x}{\sqrt[3]{1+x^2}}\,dx$ (c) $\displaystyle\int_0^\infty \frac{1}{(1+x^2)^{3/2}}\,dx$

(d) $\displaystyle\int_0^1 \frac{\ln x}{x}\,dx$ (e) $\displaystyle\int_4^5 \frac{x}{\sqrt{x-4}}\,dx$ (f) $\displaystyle\int_0^1 \frac{2}{x^2-1}\,dx$

3. Evaluate each improper integral; determine whether it converges or not.

(a) $\displaystyle\int_{-\infty}^0 xe^{2x}\,dx$ (b) $\displaystyle\int_0^\infty \frac{e^{-x}}{1+e^{-x}}\,dx$ (c) $\displaystyle\int_2^\infty \frac{1}{(x-1)^2}\,dx$

(d) $\displaystyle\int_0^2 \frac{1}{\sqrt{2-x}}\,dx$ (e) $\displaystyle\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx$ (f) $\displaystyle\int_2^4 \frac{1}{\sqrt{x^2-4}}\,dx$

4. Use #3(e) to find the average value of $\frac{1}{\sqrt{4-x^2}}$ on $[0,2)$. (Answer: $\pi/4$).

5. This is a beautiful problem which asks you to use many of the ideas of the course, including l'Hôpital's rule and improper integrals. Consider the infinitely deep well formed by rotating the region bounded by the curves curve $y = \ln x$, the y-axis, and the x-axis in the fourth quadrant. it is filled with water (density $D = 62.5$ lbs/ft^3). How much work W is done in emptying the well (raising water to ground level $H = 0$.). The formula for work is

$$W = D\int_a^b (H-y)\pi(f(y))^2\,dy$$

where a and b the depths of the bottom and top of the well and $x = f(y)$ represents the equation of the curve as a function of x. Hint: Use your work in #3(a). [Answer: 15.625π ft-lbs]

Some Answers

2. (a) $\lim_{b\to\infty} \ln\frac{b-1}{b+1} - \ln\frac{1}{3} = \ln 1 + \ln 3 = \ln 3$

(b) $\lim_{b\to\infty} \frac{3}{2}(1+b^2)^{2/3} - 3 = +\infty$. Diverges

(c) $\lim_{b\to\infty} \frac{b}{\sqrt{1+b^2}} = \lim_{b\to\infty} \frac{b}{b\sqrt{\frac{1}{b^2}+1}} = 1$

(d) $\lim_{a\to 0^+} \frac{1}{2}[(\ln 1)^2 - (\ln a)^2] = -\infty$. Diverges.

(e) $8\frac{2}{3}$.

(f) $\lim_{b\to 1^-} \ln|\frac{b-1}{b+1}| - \ln 1 = -\infty$. Diverges.

3. You may need l'Hôpital's rule.

(a) 3   (b) $-\infty$ Diverges   (c) ∞ Diverges

(d) 1   (e) 0   (f) 1   (g) $\dfrac{1}{\sqrt{e}}$

4. (a) $-\frac{1}{4}$.

(b) $\lim_{b\to\infty} -\ln(1+\frac{1}{e^b}) + \ln 2 = \ln 2$

(c) $\lim_{b\to\infty} -(\frac{1}{1-b}) - (-1) = 1$

(d) $\lim_{b\to 2^-} -2\sqrt{(2-b)} + 2\sqrt{2} = 2\sqrt{2}$.

(e) $\lim_{b\to 2^-} \arcsin\frac{b}{2} - \arcsin 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

(f) Triangle: $\ln(2+\sqrt{3})$.